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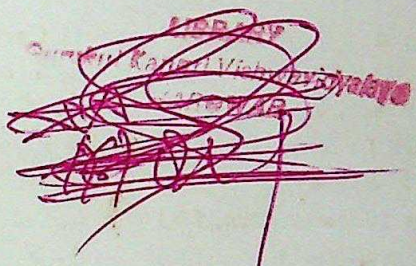

VOLUME 22

1992

Published by :

The Vijnana Parishad of India

DAYANAND VEDIC POSTGRADUATE COLLEGE

Bundelkhand University

ORAI, U. P., INDIA

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This volume of

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is being dedicated to

the memory of

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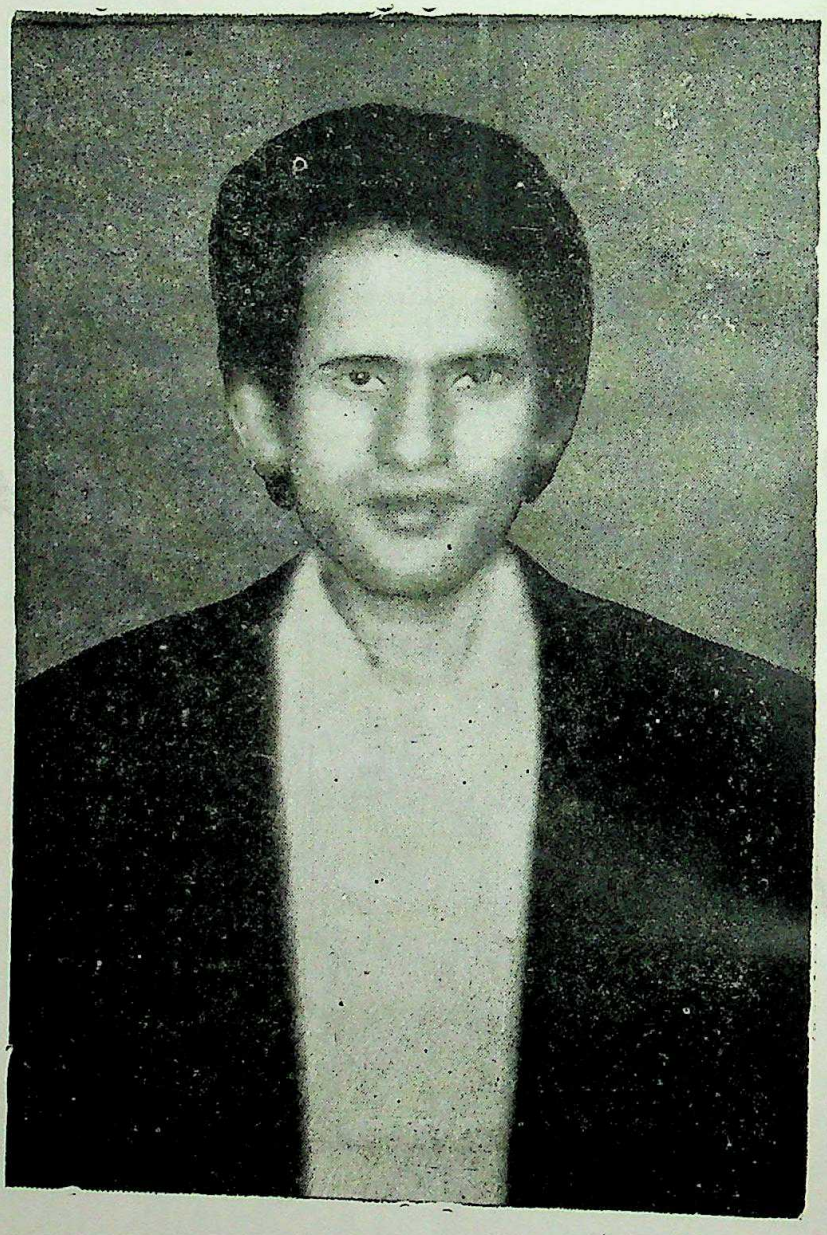

Professor Kanhaya Lal Singh

(February 15, 1944—November 22, 1990)

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

KANHAYA LAL SINGH

By

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(Received : October 10, 1992)

Dr. Kanhaya Lal Singh was born on February 15,1944 in Varanasi, U. P., India. From his childhood he was a very aggressive and determined boy.If he wanted to get some thing then it was difficult to satisfy him until he obtained it.

He was hard working, sincere and highly motivated with his studies. During his University days he developed interest in sports and concentrated on becoming a champion, and he worked very hard to achieve this goal.

In 1965 he came to Canada and was awarded M. S. degree in Mathematics in the year 1967. He taught at the University of South Carolina before going to complete his Ph.D. degree in Mathematics (Fixed point Theory)at the Texas A and M University, College Station, Texas, U.S.A..

After completing his Ph.D. he worked as a Post-Doctoral Fellow at the Lakehead University,Canada and published well over 20 papers.

During that period jobs were scarce and one was lucky to get a permanent position in a good institution. When he was offered a position in Memorial University,after *three* reminders,he finally phoned and told his brother, who was a Professor in that Department that others would think that he was given the job because of his brother and not by his merit or qualifications. He declined the offer and preferred to remain as a post-doctoral-fellow.

He was keen in accepting a position in U. S. A. as he did not like the winter of Canada. He first moved to the University of Minnesota

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at Duluth and worked as an Assistant Professor and then accepted a job of Associate Professor in the Fayetteville State University, North Carolina and remined there. Like father like son, he was a very successful teacher, always liked by his students and colleagues. He brought name and fame to the department by his hard work and devotion. He was soon promoted to the rank of Full Professor. In 1989, in recognition of outstanding professional performance he was nominated by the Department of Mathematics and Computer Science, for the Faculty of the year,1989 but unfortunately could not live and on November 22, 1990 he passed away.The mathematical community suffered a great loss and the university lost a very valuable faculty member.

He has a very impressive list of publications and his collaborators had very high regard for the quality of his work. Once a collaborator remarked that *KL is a moving dictionary of nonlinear analysis*, you ask any thing (information) and that is on his tips. Among his collaborators-Anderson, Guay, Naimpally, Rhoades, Sahney and Whitfield have several joint papers. While he was in hospital, the Head, Department of Mathematics at Duluth phoned his brother and told him that he is such a talanted mathematician we feel very sorry to see him hospitalized.

His main hobby in a way was mathematics and he used to devote his full time, energy and money for the pursuit of mathematical knowledge. A medical doctor told his brother, while he was on death bed, that KL has written a paper giving mathematical application in medicine just a few weeks ago.

We all miss his absence and pray to God to give his soul an eternal peace.

Acknowledgements

My sincere thanks are due to Professer S. P. Singh for personally communicating the news of Professor K.L. Singh's death to me, and for supplying me with the bibliography and many of the other details presented here.

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(Dedicated to the memory of Professor K. L. Singh)

A GENERAL SOLUTION OF A PARTIAL DIFFERENTIAL EQUATION OCCURRING IN THE ORNSTEIN-UHLENBECK DIFFUSION PROCESS

By

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(Received: July 26, 1991; Revised: October 31, 1991)

ABSTRACT

A general solution of the backword equation occurring in the Ornstein- Uhlenbeck diffusion process is obtained.

1. INTRODUCTION

In this note, we present a general solution of the backward equation [3, p. 155, (20)]:

$$(1\ 1)\quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - x\,\frac{\partial u}{\partial x}\ (-\infty < x < \infty),$$

which uniquely determines the behaviour of the Ornstein-Uhlenbeck (O-U) diffusion process. As a praticular case of our solution (2.4) below, we obtain the following elementary solution given by Karlin and McGregor [2, p. 170]:

$$(1.2)\quad p(t;x,y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi}\,2^{n+1}\,n!}\ e^{-nt} H_n(x/\sqrt{2})\, H_n(y/\sqrt{2}),$$

where $H_n(x)$ denotes the Hermite polynomial of degree n in x.

The backward equation (1.1) also occurs in the study of the limiting behaviour of the many-server queue [3, pp. 154–155].

2. THE SOLUTION OF THE BACKWARD PARTIAL DIFFERENTIAL EQUATION.

Let us assume that (1.1) has a solution of the form:

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(2.1) $u(x,t) = e^{-nt}\, y(x/\sqrt{2}),\ n=0,1,2,\ldots$.

The substitution of (2.1) into (1.1) yields the differential equation:

(2.2) $y''(x/\sqrt{2}) - \sqrt{2}\, xy'(x,\sqrt{2}) + 2ny\, x/\sqrt{2}) = 0,$

which is Hermite's equation [3, p. 54, (23)] with solution $y = H_n(x/\sqrt{2})$. Hence we conclude that to each eigenvalue given by (2.2), there corresponds the solution of (1.1), called an eigenfunction or eigenstate given by

(2.3) $u_n = e^{-nt}\, H_n(x/\sqrt{2}),\ n=0,1,2,\ldots$.

On using the principle of superposition, the solution of (1.1) takes the form:

(2.4) $u(x,t) = \sum_{n=0}^{\infty} C_n\, e^{-nt} H_n(x/\sqrt{2}).$

Putting t=0 in (2.4) we have

(2.5) $u(x,0) = \sum_{n=0}^{\infty} C_n\ H_n(x/2).$

Multiplying both sides of (2.5) by $e^{-x^2/2}\ H_m(x/\sqrt{2})$ and integrating with respect to x from $-\infty$ to ∞, and using the orthonality property of the Hermite polynomials [3, p. 54, (17)] after setting $x/\sqrt{2}$ for x, we obtain the Fourier-Hermite coefficients:

(2.6) $C_n = \dfrac{1}{2^n n!(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-x^2/2}\ H_n(x/\sqrt{2})u(x)\ dx.$

3. THE PARTICULAR SOLUTION

In (2.6), putting

$$u(x) = \frac{e^{y^2/4}}{2(2\pi)^{1/2}\, i^n}\ e^{x^2/4 + iyx/2},$$

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and using the following integral [1, p. 174, 7 (a)] (with $t=x/\sqrt{2}$ and $x=y/\sqrt{2}$) :

$$(3.1)\quad \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{-\lambda^2/4 + iyx/2} H_n(x/\sqrt{2})\, dx = \sqrt{2}i^n\, e^{-y^2/4}\, H_n(y/\sqrt{2})$$

we get

$$(3.2)\quad C_n = \frac{1}{\sqrt{\pi}\, 2^{n+1} n!}\, H_n(y/2).$$

Subtituting the value of C_n from (3.2) into (2.4), we obtain the elementary solution (1.2) given by Karlin and McGregor.

Remark 1. On evaluating the integral in (2.5) for a particular value of $u(x)$, the values of $C_0\ C_1\ C_2,\ldots$ for that value of $u(x)$ can be obtained. This exercise may be useful for computation.

Remark 2. In a future communication we propose to obtain the values of the Fourier-Hermite coefficients C_n for $u(x,0)$ as Meijer's G-function [3, pp. 37-39], which is a generalization of almost all special function appearing in applied mathematics, physical sciences, and statistics.

Acknowledgements

The author expresses his gratitude to Professor H. M. Srivastava for some useful suggestions for revising the paper.

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

ON SOME NEW RESULTS INVOLVING DOUBLE AND TRIPLE HYPERGEOMETRIC FUNCTIONS

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(Received: May 5, 1990; Revised [Final]: January 10, 1992)

ABSTRACT

The aim of this paper is to establish some new generating relations for certain hypergeometric functions of two and three variables.

1. INTRODUCTION

If we use the notation $(a)_n = a(a+1) \ldots (a+n-1)$, $(a)_0 = 1$, where a is arbitrary and n a positive integer, then the needed hypergeometric functions of two and three variables are defined as follows:

$$F_E(\alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x.y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p} (\beta_1)_m (\beta_2)_{n+p} x^m y^n z^p}{m!\, n!\, p!\, (\gamma_1)_m (\gamma_2)_n (\gamma_3)_p}$$

$$|x| < r, |y| < s, |z| < t, \qquad \ldots (1\ 1)$$

$$F_G(\alpha_1, \alpha_1\ \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p} (\beta_1)_m (\beta_2)_n (\beta_3)_p x^m y^n z^p}{m!\, n!\, p!\, (\gamma_1)_m (\gamma_2)_{n+p}}$$

$$|x| < 1, |y| < 1, |z| < 1, \qquad \ldots (1.2)$$

$$F_A(\alpha, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_3; x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+n+p} (\beta_1)_m (\beta_2)_n (\beta_3)_p x^m y^n z^p}{m!\, n!\, p!\, (\gamma_1)_m (\gamma_2)_n (\gamma_3)_p}$$

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$$|x| + |y| + |z| < 1, \qquad \ldots (1.3)$$

$$H_A(\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p} (\beta)_{m+n} (\beta')_{n+p} x^m y^n z^p}{(\gamma)_m (\gamma')_{n+p} m!\, n!\, p!}$$

$$|x| < r, |y| < s, |z| < t, \quad r+s+t = 1 + st, \qquad \ldots (1.4)$$

$$G_B(\alpha, \beta_1, \beta_2, \beta_3; \gamma; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m} (\beta_1)_m (\beta_2)_n (\beta_3)_p x^m y^n z^p}{(\gamma)_{n+p-m} m!\, n!\, p!}$$

$$|x| < 1, |y| < 1, |z| < 1, \qquad \ldots (1.5)$$

$${}^3G_B^{(1)} (\alpha, \beta_1, \beta_2; \gamma; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m} (\beta_1)_m (\beta_2)_n x^m y^n z^p}{(\gamma)_{n+p-m} m!\, n!\, p!} . \qquad \ldots (1.6)$$

$$\Gamma_1 (\alpha, \beta, \beta', x, y)$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\beta)_{n-m} (\beta')_{m-n} x^m y^n}{m!\, n!} . \qquad \ldots (1.7)$$

$$H_2 (\alpha, \beta, \gamma, \delta, \epsilon, x, y)$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n} (\beta)_m (\gamma)_n (\delta)_n x^m y^n}{(\epsilon)_m m!\, n!} . \qquad \ldots (1.8)$$

$$H_3 (\alpha, \beta, \gamma, x\, y)$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n} (\beta)_n x^m y^n}{(\gamma)_{m+n} m!\, n!} . \qquad \ldots (1.9)$$

(See, for details, [1], [2], [3], [5] and [8].)

In the present investigation we also require the following relations:

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$$\sum_{n=0}^{\infty} \frac{(\lambda)_n t^n}{n!} F_1(-n,\mu,\nu;\alpha;x,y) = (1-t)^{-\lambda} F_1(\lambda,\mu,\nu;\alpha;\frac{xt}{t-1},\frac{yt}{t-1})$$

$$max\left[\left|\frac{xt}{t-1}\right|, \left|\frac{yt}{t-1}\right|, |t|\right\} < 1, \qquad \ldots (1,10)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n t^n}{n!} F_2(-n,\mu,\nu;\alpha,\beta;x,y) = (1-t)^{-\lambda} F_2(\lambda,\mu,\nu;\alpha,\beta;\frac{xt}{t-1},\frac{yt}{t-1}),$$

$$max\left\{\left|\frac{xt}{t-1}\right|+\left|\frac{yt}{t-1}\right|, |t|\right\} < 1, \qquad \ldots (1.11)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n t^n}{n!} F_4(-n,\mu;\alpha,\beta;x,y) = (1-t)^{-\lambda} F_4(\lambda,\mu,\alpha,\beta;\frac{xt}{t-1},\frac{yt}{t-1}),$$

$$max\left\{|xt/(t-1)|^{1/2}+|yt/(t-1)|^{1/2}, |t|\right\} < 1, \qquad \ldots (1.12)$$

$$\sum_{n=0}^{\infty} (u-1)^n p_n^{(\alpha-n,\beta+n)}\left(\frac{1+u}{1-u}\right) {}_2F_1(-n,\delta-\gamma,\delta,t)$$

$$= \frac{\Gamma(\delta)(\delta+\alpha-\gamma)}{\Gamma(\delta+\alpha)\Gamma(\delta-\gamma)} t^{\alpha}(1-u+ut)^{-\beta} {}_2F_1\left(\beta,\gamma,\beta+\alpha,\frac{-ut}{1-u+ut}\right), \qquad \ldots (1.13)$$

where F_1, F_2, F_4 are Appell's double hypergeometric functions [2,p.224 (6), (7), (9)] and (1.10), (1.11), (1.12) are known results given by Srivastava [5, p. 86, (4.1), (4.2), (4.3)].

2. GENERATING RELATIONS

We prove here the following formulae:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} L_1(\alpha,1-\beta,\lambda+n,x,y) F_4(-n,\mu;\nu,\epsilon;u,v)t^n$$

$$= (1-t)^{-\lambda} \sum_{q=0}^{\infty} \frac{(1-\beta)_q (y(t-1))^q}{(1-\lambda)_q q!} \cdot F_E\left(\lambda-q,\lambda-q,\alpha,\mu,\mu;\beta-q,\nu,\epsilon;\frac{x}{t-1},\frac{ut}{t-1},\frac{vt}{t-1}\right),$$

$$\ldots (2.1)$$

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$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} H_2(\lambda+n,\alpha,\beta,\gamma,\delta,x,y)\, F_1(-n,\mu,\nu;\epsilon;u,v)t^n$$

$$= (1-t)^{-\lambda} \sum_{q=0}^{\infty} \frac{(\beta)_q(\gamma)_q(y(t-1))^q}{q!\,(1-\lambda)_q},$$

$$F_G\left(\lambda-q, \lambda-q, \lambda-q, \alpha, \mu, \nu; \delta, \epsilon, \epsilon; \frac{x}{1-t}, \frac{ut}{t-1}, \frac{vt}{t-1}\right); \qquad \ldots (2.2)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} H_3(\lambda+n, \beta, \gamma, x, y, F_2(-n,\mu,\nu;\alpha,\epsilon;u,v,)t^n$$

$$= (1-t)^{-\lambda} \sum_{p=0}^{\infty} \frac{(\lambda)_{2p}}{(\gamma)_p\, p!} \left(\frac{x}{(1-t)^2}\right)^p$$

$$F_A\left(\lambda+2p,\beta,\mu,\nu;\gamma+p,\alpha,\epsilon; \frac{\lambda}{1-t}, \frac{ut}{t-1}, \frac{vt}{t-1}\right). \qquad \ldots (2\ 3)$$

To prove (2.1), consider:

$$\phi = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \Gamma_1(\alpha, 1-\beta, \lambda+n, x, y)\, F_4(-n,\mu;\nu,\epsilon; u, v)t^n$$

Express Γ_1 in series form as given in (1.7), employ

$$(\lambda)_n (\lambda+n)_{p-q} = (\lambda)_{n+p-q} = (\lambda)_{p-q} (\lambda+p-q)_n, \qquad \ldots (2.4)$$

and apply relation (1.12); we find that

$$\phi = \sum_{p,q=0}^{\infty} \frac{(\alpha)_p (1-\beta)_{q-p} (\lambda)_{p-q} x^p y^q}{p!\ q!}$$

$$F_4(\lambda+p-q, \mu; \nu, \epsilon; \left(\frac{ut}{t-1}, \frac{vt}{t-1}\right) \cdot (1-t)^{-(\lambda+p-q)}.$$

Writing F_4 in series form, we have

$$\phi = (1-t)^{-\lambda} \sum_{q=0}^{\infty} \frac{(\lambda)_{-q} (1-\beta)_q (y(1-t))^q}{q!}$$

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$$\sum_{r,s,p=0}^{\infty} \frac{(\lambda-q)_{p+r+s}\,(\mu)_{r+s}\,(\alpha)_p}{(\beta-q)_p\,(\nu)_r\,(\epsilon)_s\,r!\,p!\,s!} \left(\frac{x}{t-1}\right)^p \left(\frac{ut}{t-1}\right)^r \left(\frac{vt}{t-1}\right)^s$$

which in the light of (1.1) provides (2.1).

The derivations of formulas (2.2), and (2.3) would run parallel to what we have obtained above.

3. GENERATING RELATIONS

In this section the following generating relations have been established:

$$(1-z)^{\beta}\,(1-y)^{\gamma}\,H_A\left(\beta,\gamma,\beta';\delta+\alpha,\beta';\frac{-xt(1-z)\,(1-y)}{(1-x+xt)},y,z\right)$$

$$= \frac{\Gamma(\delta+\alpha)\,\Gamma(\delta-\gamma)}{\Gamma(\delta)\,\Gamma(\delta+\alpha-\gamma)}\,t^{-\alpha}\,(1-x+xt)^{\beta}\sum_{n=0}^{\infty}\frac{n!}{(\delta)_n}\,(x-1)^n\,p_n^{(\alpha-n,\,\beta+n)}\left(\frac{1+x}{1-x}\right)$$

$$\cdot\,p_n^{(\delta-1,\,-\gamma-n)}\,(1-2t)\,, \qquad \ldots (3.1)$$

$$(1-z)^{\beta_3}\,G_B\,(\alpha,\beta_1,\beta_2,\beta_3;\alpha;x,x(1-y)/2\,,z)$$

$$= \sum_{m=0}^{\infty}(-x)^m\,p_m^{(-m-\beta_1,\beta_2+\beta_1-1)}\,(y). \qquad \ldots (3.2)$$

$${}_3G_B^{(1)}\,(\alpha,\beta,\gamma;\alpha;x,x(1-y)/2,\,z)$$

$$= exp\,(x)\sum_{m=0}^{\infty}(-x)^m\,p_m^{(-\beta-m,\,\gamma+\beta-1)}\,(y). \qquad \ldots (3\,3)$$

To prove (3.1), let us consider:

$$T = (1-z)^{\beta}\,(1-y)^{\gamma}\,H_A\left(\beta,\gamma,\beta';\delta+\alpha,\beta';\frac{-xt\,(1-z)\,(1-y)}{(1-x+xt)},y,z\right)$$

and express H_A in series form, applying the result

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$$(1-z)^{-a} = \sum_{i=0}^{\infty} \frac{(a)_i z^i}{i!}, \qquad \ldots (3.4)$$

we get

$$T = {}_2F_1\left(\beta, \gamma, \delta+\alpha; \frac{-xt}{(1-x+xt)}\right),$$

which in the light of (1.13) provides (3.10).

The proofs of the results (3.2) and (3.3) are similar to that of (3.1).

4. PARTICULAR CASES

In (2.1) putting $\nu = 0$ and simplifying we obtain:

$$\sum_{p,q=0}^{\infty} \frac{(\alpha)_p (\lambda)_{p-q} (1-\beta)_{q-p} \left(\frac{x}{1-t}\right)^p (y(1-t))^q}{p!\, q!} \; {}_2F_1\left(\lambda+p-q, \mu; \nu; \frac{ut}{t-1}\right)$$

$$= \sum_{q=0}^{\infty} \frac{(1-\beta)_q (y(t-1))^q}{q!\,(1-\lambda)_q} F_2\left(\lambda-q, \alpha, \mu; \beta-q, \nu; \frac{x}{t-1}, \frac{ut}{t-1}\right), \qquad \ldots (4.1)$$

In (3.1) letting $z=0$ and simplifying, we get

$${}_2F_1\left(\beta, \nu, \delta+\alpha, \frac{-xt}{1-x+xt}\right) = \frac{\Gamma(\delta+\alpha)\Gamma(\delta-\nu)}{\Gamma(\delta)\Gamma(\delta+\alpha-\nu)} t^{-\alpha} (1-x+xt)^{\beta}$$

$$\sum_{n=0}^{\infty} \frac{(x-1)^n n!}{(\delta)_n} P_n^{(\alpha-n, \beta+n)}\left(\frac{1+x}{1-x}\right) \cdot P_n^{(\delta-1, -\gamma-n)}(1-2t). \qquad \ldots (4.2)$$

In (3.2) taking $z=0$ replacing $1-y$ by $\frac{1-y}{\beta_2}$ and letting $\beta_2 \to \infty$ we obtain:

$$(1-x)^{-\beta_1} \sum_{n=0}^{\infty} \frac{\left[x\left(\frac{1-y}{2}\right)\right]^n}{n!} = \sum_{m=0}^{\infty} (-x)^m L_m^{(-\beta_1-m)}\left(\frac{1-y}{2}\right). \qquad \ldots (4.3)$$

In (3.3) changing $1-y$ by $\frac{1-y}{\gamma}$ and putting $\gamma \to \infty$, we obtain

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$$exp\,(z)\,(1-x)^{-\beta}\sum_{n=0}^{\infty}\frac{\left[x\left(\frac{1-y}{2}\right)\right]^n}{n!}=exp(z)\sum_{m=0}^{\infty}(-x)^m L_m^{(-\beta-m)}\left(\frac{1-y}{2}\right),$$

Acknowledgements

The author is indebted to Dr. B. M. L. Shrivastava for guidance and to Prof. H. M. Srivastava of the University of Victoria, Canada, for his valuable suggestions for improvement of the paper.

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

UNIFIED PRESENTATION OF TWO GENERAL SEQUENCES OF FUNCTIONS

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(Received: November 29, 1989)

ABSTRACT

In the present paper, an unified study of two general sequences of functions defined by Bhargava [1] and Chandel-Agrawal [4], has been made through operational techniques.

1. INTRODUCTION

To unify the study two general classes of functions due to Chandel and Bhargava [2] and Singh [5], Bhargava [1] introduced a sequence of functions defined by :

$$(1.1)\quad G_n\{a, k, p; g(x), h(x)\} = e^{-p\,g(x)}\, T^n_{a,k}\left\{h^n(x),\ e^{p\,g(x)}\right\},$$

where $T_{a,k} = x^k(a + xD)$; $h(x)$, $g(x)$ are suitable functions of x and a, k, p are independent of x.

Recently, to unify the study of two general classes of functions of Chandel-Bhargava [2] and Chandel-Agrawal[3]. Chandel and Agarwal [4] introduced a sequence of functions defined by :

$$(1.2)\quad G_n^{(a,b,k)}(h, g(x)) = [1-h\,g(x)]^b\, T^n_{a,k}\left\{(1-h\,g(x))^{-b}\right\},$$

where a, b, h, k are independent of x and $g(x)$ is a suitable differentiable function of x.

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The motivation of this paper is to unify the study of two general saquences of functions defined by (1.1) and (1.2), by introducing the sequence of functions

$$\left\{ R_n^{(a,b,k,p)} (h(x), g(x)) / n=0,1,2, \ldots \right\}$$

defined by Rodrigues' formula

$$(1.3) \quad R_n^{(a,b,k,p)} (h(x), g(x)) = [1-pg(x)]^b T_{a,k}^n \left\{ (h(x))^n [1-pg x)]^{-b} \right\},$$

where a b, k, p are independent of x and $h(x)$, $g(x)$ are suitable differentiable functions of x.

Particularly replacing p by p/b and $b \to \infty$, (1.3) reduces to (1.1). For $h(x)$ independent of x and $p=h$, (1.3) reduces to (1.2).

2. Operational formulas. Consider

$$[1-pg(x)]^b T_{a,k}^n \left\{ (h(x))^n [1-p g(x)]^{-b} f \right]$$

$$= [1-p g(x)]^b \sum_{s=0}^{n} \binom{n}{s} T_{a,k}^{n-s} \left\{ (h(x))^n [1-p g(x)]^{-b} \right\} T_{0,k}^s \{f\}$$

$$= [1-p g(x)]^b \sum_{s=0}^{n} \binom{n}{s} T_{0,k}^{n-s} \left\{ [(h(x))^{n/(n-s)}]^{n-s} [1-pg(x)]^{-b} \right\} T_{0,k}^s \{f\}$$

Therefore

$$(2.1) \quad [1-p g(x)]^b T_{a,k}^n \left\{ (h(x))^n [1-pg(x)]^{-b} f \right\}$$

$$= \sum_{s=0}^{n} \binom{n}{s} R_{n-s}^{(a,b,k,p)} \left((h(x))^{n/(n-s)}, g(x) \right) T_{0,k}^s \{f\}.$$

Again considering

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$$[1-pg(x)]^b \; T^n_{a,k}\left\{ (h(x))^n \; [1-pg(x))^{-b} \; f \; \right\}$$

$$= [1-pg(x)]^b \; T^{n-1}_{a,k}\left\{ (ax^k + x^{k+1}D) \; [(h(x))^n \; (1-pg(x))^{-b} \; f \;]\right\}$$

$$= [1-pg(x)]^b \; T^{n-1}_{a,k}\left\{ (h(x))^n \; (1-pg(x))^{-b} \right.$$

$$\left. \left[x^k \; (a+xD) + x^{k+1}\left(\frac{nh'(x)}{h(x)} + \frac{pbg'(x)}{1-pg(x)}\right)\right]\right\} f$$

$$= [1-pg(x)]^b \; ((h(x)^n \; [1-pg(x)]^{-b} \; \left[T_{a,k} + x^{k+1}\left(\frac{nh'(x)}{h(x)} + \frac{pbg'(x)}{1-pg(x)}\right)\right]^n f,$$

we obtain

$$(2.2 \quad [1-pg(x)]^b \; T^n_{a,k} \; \{(h(x))^n \; (1-pg(x))^{-b} \; f\}$$

$$= (h(x))^n \left[T_{a,k} + x^{k+1} \left(\frac{nh'(x)}{h(x)} + \frac{pbg'(x)}{1-pg(x)}\right)\right]^n f.$$

From (2.1) and (2.2), we derive

$$(2.3) \quad ((h(x))^n \left[T_{a,k} + x^{k+1} \left(\frac{nh'(x)}{h(x)} + \frac{pbg'(x)}{1-pg(x)}\right)\right]^n f$$

$$= \sum_{s=0}^{n} \binom{n}{s} \; R^{(a,b,k,p)}_{n-s} \left((h(x)^{n/(n-s)}, \; g(x)\right) \; T^s_{0,k} \; \{f\},$$

which particularly, for $f(x) = 1$, further gives

$$(2.4) \left[T_{a,k} + x^{k+1} \left(\frac{nh'(x)}{h(x)} + \frac{pbg'(x)}{1-pg(x)}\right)\right]^n \{1\}$$

$$= (h(x))^{-n} \; R^{(a,b,k,p)}_{n} \; (h(x)^{n/(n-s)}, g(x)).$$

Next consider

$$T^n_{a,k}\left\{ (h(x))^n \; [1-pg(x)^{-b} \; f \right\}$$

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$$= T_{a,k}^{n-1}\left\{(h(x)^{n-1}(1-pg(x))^{-b}\left[h(x)\,T_{a,k} + x^{k+1}\left(nh'(x)+\frac{pbg'(x)h(x)}{1-pg(x)}\right)\right] f\right\}$$

$$= T_{a,k}^{n-2}\left\{(h(x))^{n-2}(1-pg(x))^{-b}\left[h'x)\,T_{a,k}+x^{k+1}\left((n-1)h'(x)+\frac{pbh(x)g'(x)}{1-pg(x)}\right)\right]\right.$$

$$\left.\left[h(x)\,T_{a,k} + x^{k+1}\left(nh'(x) + \frac{pbh(x)g'(x)}{1-pg(x)}\right)\right] f\right\}$$

$$= h(x)^{n-n}(1-pg(x))^{-b}\left[h(x)\,T_{a,k} + x^{k+1}\left(n-(n-1)h'(x) + \frac{bpg'(x)\,h(x)}{1-pg(x)}\right)\right]$$

$$\ldots\left[h(x)\,T_{a,k} + x^{k+1}\,(nh'(x) + \frac{pb\,h(x)\,g'(x)}{1-pg(x)}\right]f.$$

Therefore

$$(2.5)\quad T_{a,k}^{n}\left\{(h(x))^{n}\,[1-pg(x)]^{-b}\,f\right\}$$

$$= [1-pg(x)]^{-b}\prod_{j=0}^{n-1}\left[h\,T_{a,k} + x^{k+1}\left((n-j)h'(x) + \frac{pbh(x)\,g'(x)}{1-pg(x)}\right)\right]f.$$

Making an appeal to (2.1) the above result (2.5) gives

$$(2.6)\quad \sum_{s=0}^{n}\binom{n}{s}\,R_{n-s}^{(a,b,k,p)}\left((h(x))^{n/(n-s)},g(x)\right)\,T_{0,k}^{s}\,\{f\}$$

$$= \prod_{j=0}^{n-1}\left[h(x)\,T_{a,k} + x^{k+1}\left\{(n-j)\,h'(x) + \frac{pb\,h(x)g'(x)}{1-pg(x)}\right\}\right]\{f\},$$

which particularly for $f(x) = 1$, further gives

$$(2.7)\quad \prod_{j=0}^{n-1}\left[h(x)\,T_{a,k} + x^{k+1}\left\{(n-j)\,h'(x) + \frac{pbh(x)\,g'(x)}{1-pg(x)}\right\}\right]\{1\}$$

$$= R_{n}^{(a,b,k,p)}(h(x),\,g(x)).$$

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Again consider

$$T^{n}_{a,k}\left\{(h(x))^{n}[1-pg(x)]^{-b}f\right\}$$

$$=T^{n-1}_{a,k}\left\{(h(x))^{n-1}[1-pg(x)]^{-b}\left[x^{k+1}\left(nh'(x)+\frac{pbh(x)g'(x)}{1-pg(x)}\right)+h(x)T_{a,k}\right]f\right\}$$

$$=T^{n-1}_{a,k}\left\{(h(x))^{n-(m+1)}[1-pg(x)]^{-b}\left[x^{k+1}\left\{(n-m-1+(m+1)(h(x))^{m}h'(x)\right.\right.\right.$$

$$\left.\left.\left.+\frac{(h(x))^{m+1}pbg'(x)}{1-pg(x)}\right\}+(h(x))^{m+1}T_{a,k}\right]f\right\}$$

$$=(h(x))^{-nm}[1-pg(x)]^{-b}\prod_{j=1}^{n}\left[x^{k+1}\left\{(n-(m+1)+(m+1))(h(x))^{m}h'(x)\right.\right.$$

$$\left.\left.+\frac{(h(x))^{m+1}pbg'(x)}{1-pg(x)}\right\}+(h(x))^{m+1}T_{a,k}\right]f.$$

Thus

$$(2.8)\quad T^{n}_{a,k}\left\{(h(x))^{n}[1-pg(x)]^{-b}f\right\}$$

$$=(h(x))^{-nm}[1-pg(x)]^{-b}\prod_{j=0}^{n-1}\left[x^{k+1}\left\{(n-(m+1)j)(h(x))^{m}h'(x)+\right.\right.$$

$$\left.\left.\frac{(h(x))^{m+1}pbg'(x)}{1-pg(x)}\right\}+(h(x))^{m+1}T_{a,k}\right]f,$$

which by making an appeal to (2.1), gives

$$(2.9)\quad (h(x))^{nm}\sum_{s=0}^{n}\binom{n}{s}R^{(a,b,k,p)}_{n-s}((h(x))^{n/(n-s)},g(x))\,T^{s}_{0,k}\{f\}$$

$$=\prod_{j=0}^{n-1}\left[(h(x))^{m+1}T_{a,k}+x^{k+1}\{n-(m+1)j\}(h(x))^{m}h'(x)\right.$$

$$\left.+\frac{(h(x))^{m+1}pbg'(x)}{1-pg(x)}\right]f.$$

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Particularly for $f(x) = 1$, from above result, we derive

$$(2.10)\quad \prod_{j=0}^{n-1}\left[(h(x))^{m+1}\,T_{a,k}+x^{k+1}\left\{\frac{(h(x))^{m+1}}{1-pg(x)}\frac{pbg'(x)}{}\right.\right.$$

$$\left.\left.+\,(n-(m+1)j)\,(h(x))^{m}h'(x)\right\}\right]\{1\} = (h(x))^{nm}R_{n}^{(a,b,k,p)}\,(h(x),\,g(x)).$$

Further consider

$$T_{a,k}^{n}\left\{(h(x))^{n}\,[1-pg(x)]^{-b}\,f\right\}$$

$$= T_{a,k}^{n-1}\left\{(h(x))^{n}[1-pg(x)]^{-b}\,x^{-m}\left[\,x^{m+k+1}\left(\frac{nh'(x)}{h(x)}+\frac{pbg'(x)}{1-pg(x)}\right)+\,x^{m}T_{a,k}\right]f\right\}$$

$$= T_{a,k}^{n-2}\left\{(h(x))^{n}x^{-2m}\,[1-pg(x)]^{-b}\left[\,x^{m+k+1}\left(\frac{nh'(x)-mh(x)/x}{h(x)}+\frac{pbg'(x)}{1-pg(x)}\right)\right.\right.$$

$$\left.\left.+\,x^{m}\,T_{a,k}\right]\left[\,x^{m+k+1}\left(\,n\,\frac{h'(x)}{h(x)}+\frac{pbg'(x)}{1-pg(x)}\right)+\,x^{m}\,T_{a,k}\right]f\right\}.$$

Finally, we derive

$$(2.11)\quad T_{a,k}^{n}\left\{(h(x))^{n}\,[1-pg(x)]^{-b}\,f\,\right\}$$

$$= (h(x))x^{-nm}\,[1-pg(x)]^{-b}\prod_{j=1}^{n}\left[x^{m+k+1}\left(\frac{nh'(x)}{h(x)}+\frac{pbg'(x)}{1-pg(x)}\right)\right.$$

$$\left.+\,x^{m}\,T_{a,k}-m\,(j-1)\,x^{m+k}\right]f,$$

which by making an appeal to (2.1), gives

$$(2.12)\quad (h(x))^{n}x^{-nm}\prod_{j=1}^{n}\left[\,x^{m+k+1}\left(\frac{nh'(x)}{h(x)}+\frac{pbg'(x)}{1-pg(x)}\right)+\,x^{m}\,T_{a,k}\right.$$

$$\left.-\,m(\,j-1)\,x^{m+k}\right]f$$

$$= \sum_{s=0}^{n}\binom{n}{s}\,R_{n-s}^{(a,b,k,p)}\left((h(x))^{n/(n-s)},g(x)\right)\,T_{0,k}^{s}\,\{f\}$$

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Particularly for $f(x) = 1$, the above result further gives

$$(2.13)\quad (h(x))^n x^{-nm} \prod_{j=1}^{n} \left[x^{m+k+1} \left(\frac{nh'(x)}{h(x)} + \frac{pbg'(x)}{1-pg(x)} \right) + x^m T_{a,k} - m(j-1) x^{m+k} \right] \{1\}$$

$$= R_n^{(a,b,k,p)} (h(x), g(x)).$$

3. Other relations Consider

$$T_{a,k}^s \left\{ R_n^{(a,b,k,p)} (h(x), g(x)) \right\}$$

$$= T_{a,k}^s \left[[1-pg(x)]^{-b} T_{a,k}^n \left\{ (h(x))^n [1-pg(x)]^{-b} \right\} \right]$$

$$= \sum_{j=0}^{s} \binom{s}{j} T_{a,k}^{n+s-j} \left\{ (h(x))^n [1-pg(x)]^{-b} \right\} T_{0,k}^{j} \left\{ [1-pg(x)]^b \right\}.$$

Therefore, we derive

$$(3.1)\quad T_{a,k}^s R_n^{(a,b,k,p)} (h(x), g(x))$$

$$= \sum_{j=0}^{s} \binom{s}{j} R_{n+s-j}^{(a,b,k,p)} \left((h(x))^{n/(n+s-j)}, g(x) \right) R_j^{(0,-b,k,p)} (1, g(x)).$$

Particularly for $s = 1$, we further derive

$$(3.2)\quad T_{a,k} \left\{ R_n^{(a,b,k,p)} (h(x), g(x)) \right\}$$

$$= R_{n+1}^{(a,b,k,p)} ((h(x))^{n/(n+1)}, g(x)) + R_n^{(a,b,k,p)} (h(x), g(x)) R_1^{(0,-b,k,p)} (1, g(x)).$$

Since

$$R_1^{(0,-b,k,p)} (1, g(x)) = - \frac{bpg'(x) x^{k+1}}{1-pg(x)},$$

Therefore, (3.2) further gives

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$$(3.3)\quad \left(T_{a,k} + \frac{bpg'(x)\, x^{k+1}}{1-pg(x)} \right) R_n^{(a',b',k',p)} \left(h(x),\, g(x) \right)$$

$$= R_{n+1}^{(a',b',k',p)} \left(h(x))^{n/(n+1)}, g(x) \right).$$

By iteration we derive

$$(3.4)\quad \left(T_{a,k} + \frac{bpg'(x)\, x^{k+1}}{1-pg(x)} \right)^r \left\{ R^{(a',b',k',p)} \left(h(x),\, g(x) \right) \right\}$$

$$= R_{n+r}^{(a',b',k',p)} \left((h(x))^{n/(n+r)},\, g(x) \right).$$

For brevity, we write

$$(3.5)\quad T_{a,k} + \frac{bpg'(x)\, x^{k+1}}{1+pg(x)} = \circledcirc .$$

Therefore, we have

$$(3.6)\quad \circledcirc^r \left\{ R_n^{(a',b',k',p)} \left(h(x)\,,\, g(x) \right) \right\}$$

$$= R_{n+r}^{(a',b',k',p)} \left((h(x))^{n/(n+r)}\,,\, g(x) \right) .$$

By making an appeal to (3.1) anb (3.6), we derive

$$(3.7)\quad T_{a,k}^s \left\{ R_n^{(a',b',k',p)} \left(h(x)\,,\, g(x) \right) \right\}$$

$$= \sum_{j=0}^{s} \binom{s}{j} R_j^{(0',b',k',p)} \left(1, g(x) \right) \circledcirc^{s-j} \left\{ R_n^{(a',b',k',p)} \left(h(x)\,,\, g(x) \right) \right\}$$

Further an appeal to (3.6) and the operational formula due to Chandel and Agrawal [4, (4.6)]

$$e^{t\circledcirc} \{f(x)\} = \frac{[1-pg(x)]^b}{(1-x^k k t)^{a/k}} \left[1-pg \left(\frac{x}{(1-x^k k t)^{1/k}} \right) \right]^{-b}$$

$$f \left(\frac{x}{(1-x^k k\, t)^{1/k}} \right).$$

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shows that

$$(3.8)\quad \frac{[1-pg(x)]^b}{[1-x^k t]^{a/k}}\left[1-pg\left(\frac{x}{(1-x^k t)^{1/k}}\right)\right]^{-b} R_n^{(a,b,k,p)}\left[h\left(\frac{x}{(1-x^k t)^{1/k}}\right), g\left(\frac{x}{(1-x^k t)^{1/k}}\right)\right]$$

$$= \sum_{r=0}^{\infty} R_{n+r}^{(a,b,k,p)}\left((h(x))^{n/(n+r)}, g(x)\right)\frac{t^r}{r!}.$$

Replace t by t/x^k, we obtain

$$(3.9)\quad \frac{[1-pg(x)]^b}{(1-t)^{a/k}}\left[1-pg\left(\frac{x}{(1-t)^{1/k}}\right)\right]^{-b} R_n^{(a,b,k,p)}\left[h\left(\frac{x}{(1-t)^{1/k}}\right), g\left(\frac{x}{(1-t)^{1/k}}\right)\right]$$

$$= \sum_{r=0}^{\infty} R_{n+r}^{(a,b,k,p)}\left((h(x))^{n/(n+r)}, g(x)\right)\frac{t^r}{r!\,x^{kr}\,k^r}.$$

Particularly for $n=0$, we derive

$$(3.10)\quad \frac{[1-pg(x)]^b}{(1-t)^{a/k}}\left[1-pg\left(\frac{x}{(1-t)^{1/k}}\right)\right]^{-b} = \sum_{r=0}^{\infty} R_r^{(a,b,k,p)}\left(1, g(x)\right)\frac{t^r}{r!\,k^r\,x^{kr}}.$$

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

BINOMIAL ANALOGUES OF THE CLASS OF ADDITION THEOREMS OF SRIVASTAVA, LAVOIE AND TREMBLAY

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(Received: June 4, 1987; Revised: January 16, 1989)

ABSTRACT

In the present paper, some binomial analogues of the class of addition theorems of Srivastava, Lavoie and Tremblay [8] are derived and a number of interesting applications of main results are given.

1. INTRODUCTION

Srivastava [6] presented certain interesting classes of generating functios in the forms:

$$\sum_{n=0}^{\infty} f_n^{\alpha+\lambda n}(x+ny)\,\frac{t^n}{n!}$$

and

$$\sum_{n=0}^{\infty} g_n^{(\alpha_1+\lambda_1 n,\ldots,\alpha_s+\lambda_s n)}(x_1+ny_1,\ldots,x_s+ny_s)\,\frac{t^n}{n!}$$

where

$$\left\{ f_n^{(\alpha)}(x) \right\}_{n=0}^{\infty} \quad \text{and} \quad \left\{ g_n^{(\alpha_1,\ldots,\alpha_s)}(x_1,\ldots,x_s) \right\}_{n=0}^{\infty}$$

are general one and many-parameter sequences of functions. Further, Srivastava, Lavoie and Tremblay [8] derived some general addition formulas for analogous sequences of functions and gave a number of interesting applications of the main results. Recently, Chandel and Sahgal [3] also gave a number of interesting applications and extensions of the addition theorems of srivatava, Lavoie and Tremblay [8]. In the present paper, we derive some binomial analogues of the class of addition theorems of Srivastava, Lavoie and Tremblay [8] and give a number of interesting applications of main results.

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2. Some Theorems. In this section, we establish following theorems:

Theorem 1. *Let the functions* $B(z)$ *and* $z^{-1}\,C(z)$ *be analytic in the neighbourhood of the origin, and assume (for the sake of simplicity) that*

$$B(0) = C'(0) = 1. \tag{2.1}$$

Define the sequence of functions $\left\{ G_n^{(\alpha,\beta)}(x) \right\}_{n=0}^{\infty}$ *by means of*

$$[B(z)]^{\alpha}\,[1-xC(z)]^{-\beta} = \sum_{n=0}^{\infty} G_n^{(\alpha,\beta)}(x)\,\frac{z^n}{n!}, \tag{2.2}$$

where α, β *and* x *are arbitrary complex numbers independent of* z. *Then for arbitrary parameters* λ *and* μ *independent of* z,

$$G_n^{(\alpha+\lambda\gamma,\beta+\mu\gamma)}(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{n+\gamma}{k+\gamma}\right) G_k^{(\alpha-\lambda k,\ \beta-\mu k)}(x)\; G_{n-k}^{(\gamma(k+\lambda),\ \mu(k+\gamma))}(x), \tag{2.3}$$

provided that $Re\,(\gamma) > 0$.

Theorem 2. *Let the functions* $A(z)$, $B(z)$, $z^{-1}\,C(z)$ *and* $z^{-1}\,D(z)$ *be analytic about the origin such that*

$$A(0) = B(0) = C'(0) = D'(0) = 1. \tag{2.4}$$

and define the sequence of functions $\left\{ G_n^{(\alpha,\beta,\gamma,\lambda,\mu)}(x,y,u,v) \right\}_{n=0}^{\infty}$ *by means of*

$$\sum_{n=0}^{\infty} G_n^{(\alpha,\beta,\gamma,\lambda,\mu)}(x,y,u,v)\,\frac{z^n}{n!} = [A(z)]^{\alpha}\,[B(z)]^{\beta}\,[\Omega(z)]^{\gamma}\,[1-xC(z)]^{-\lambda}\,[1-yD(z)]^{-\mu}, \tag{2.5}$$

where $\alpha, \beta, \gamma, \lambda, \mu, u, v, x$ *and* y *are independent of* z *and (for convenience)*

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$$(2.6)\quad \Omega(z) = \left[\, 1 + \frac{zuB'(z)}{B(z)} + \frac{zvxC'(z)}{1-xC(z)} \,\right].$$

Then

$$(2.7)\quad G_n^{(\alpha_1+\alpha_2,\ \beta_1+\beta_2,\ \gamma_1+\gamma_2+1,\ \lambda_1+\lambda_2,\ \mu_1+\mu_2)}(x,y,u,v)$$

$$= \sum_{k=0}^{n} \binom{n}{k} G_k^{(\alpha_1,\ \beta_1-uk,\ \gamma_1+1,\ \lambda_1-vk,\ \mu_1)}(x,y,u\ v)$$

$$G_{n-k}^{(\alpha_2,\ \beta_2+uk,\ \gamma_2+1,\ \lambda_2+vk,\ \mu_2)}(x,y,u,v)\,.$$

This theorem may be regarded as a generalization of the Theorem 1. More generally, we have

Theorem 3. *Let the function $A(z)$, $B(z)$ and $z^{-1}\,C_i(z)$ be analytic about the origin such that*

$$(2.8)\quad A(0) = B(0) = C'_i(0) = 1.\quad i=1,\ldots,m$$

and define the sequence of functions

$$\left\{ G_n^{(\alpha,\beta,\gamma,\lambda_1,\ldots,\lambda_m)}\quad (x_1,\ldots,x_m,\ u,v_1,\ldots,v_m) \right\}_{n=0}^{\infty} \text{ by means of}$$

$$(2\ 9)\quad \sum_{n=0}^{\infty} G_n^{(\alpha,\beta,\gamma,\lambda_1,\ldots,\lambda_m)}(x_1,\ldots,x_m,\ u,v_1,\ldots,v_m)\frac{z^n}{n!}$$

$$= [A(z)]^{\alpha}\ [B(z)]^{\beta}\ [\Omega(z)]^{\gamma}\ [\,1-x_1\ C_1(z)\,]^{-\lambda_1}\ \ldots\ [\,1-x_m\ C_m(z)\,]^{-\lambda_m}$$

where $\lambda_1,\ldots,\lambda_m$, $x_1,\ldots,x_m$, $v_1,\ldots,v_m$, u, α, β and γ are all arbitrary complex numbers independent of z, and (for convenience)

$$(2.10)\quad \Omega(z) = 1 + z\left[\frac{uB'(z)}{B(z)} + \frac{v_1\,x_1\,C'_1(z)}{1-x_1C_1(z)} + \ldots + \frac{v_mx_mC'_m(z)}{1-x_mC_m(z)}\right].$$

Then

$$(2.11)\quad G_n^{(\alpha+\alpha',\ \beta+\beta',\ \gamma+\gamma'+1,\ \lambda_1+\mu_1,\cdots,\lambda_m+\mu_m)}(x_1,\ldots,x_m,\ u,\ v_1,\ldots,v_m)$$

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$$= \sum_{k=0}^{n} \binom{n}{k} G_{k}^{(\alpha,\ \beta-ku,\gamma+1,\lambda_1-k\nu_1,\ldots,\lambda_m-k\mu_m)}(x_1,\ldots,x_m,\ u,\nu_1,\ldots,\nu_m)$$

$$G_{n-k}^{(\alpha',\ \beta'+ku,\ \gamma'+1,\ \mu_1+k\nu_1,\ldots,\mu_m+k\nu_m)}(x_1,\ldots,x_m,\ u,\ \nu_1,\ldots,\nu_m).$$

3. Proof of Theorem 3. Following the techniques applied for tne proof of addition formula due to Srivastava, Lavoie and Tremblay [8, p. 440, (1.7)], we can prove Theorem 3. For highlighting its connections with the earlier results, we may also refer to Srivastava and Manocha [9, Chapter 7, Problem 30].

4. Applications of Theorem 1. Chandel and Chandel [3] have given the following interesting special case for their polynomials

$$(1-pz^q)^{-c}\left[1-\frac{xz}{(1-pz^q)^r}\right]^{-b} = \sum_{n=0}^{\infty} g_n^{(b,c)}(x,\ p,\ q,\ r)\ z^n.$$

Thus for $\alpha=c$, $\beta=b$, $B(z) = (1-pz^q)^{-1}$ and $C(z) = \dfrac{z}{(1-z^q)^r}$,

the Theorem 1 gives

$$(4.1)\quad g_n^{(b+\lambda\gamma,\ c+\mu\gamma)}(x,\ p,\ q,\ r)$$

$$= \sum_{k=0}^{n} \left(\frac{n+\gamma}{k+\gamma}\right) g_k^{(b-\lambda k,\ c-\mu k)}(x,\ p,\ q,\ r)\ g_{n-k}^{(\lambda(k+\gamma),\ \mu(k+\gamma))}(x,\ p,\ q,\ r).$$

For $s = 1$, the special case of the polynomials due to Chandel and Yadava [4], is defined as

$$[a_0+a_1x_1z+\ \ldots\ +a_mx_m\ z^m]^p\left[1-\frac{r^r\ x_1\ z}{(a_0+a_1x_1z+\ldots+a_mx_m\ z^m)^r}\right]^{-q}$$

$$= \sum_{n=0}^{\infty} E_{n,m,\ p,q,r,\ 1}^{a_0,\ldots,a_m}\begin{bmatrix} x_1\ ,\ x_t \\ \vdots \\ x_m \end{bmatrix} z^m,$$

so that in the Theorem 1, taking $\alpha=p$, $\beta=q$,

$B(z)=a_0+a_1x_1z+\ldots+a_mx_mz^m$ and

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$$C(z) = \frac{r^r z}{(a_0 + a_1 x_1 z + \cdots + a_m x_m z^m)^r},$$

we get

$$(4.2)\quad E^{a_0,\ldots,a_m}_{n,\, m,\, p+\lambda\gamma,\, q+\mu\gamma,\, r,\, l}\begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix} =$$

$$= \sum_{k=0}^{n} \binom{n+\gamma}{k+\gamma} E^{a_0,\ldots,a_m}_{k,\, p-\lambda k,\, q-\mu k,\, r,\, l}\begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix}$$

$$\cdot E^{a_0,\ldots,a_m}_{n-k,\lambda(k+\gamma),\, \mu(k+\gamma),\, r.\, l}\begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix}, \quad Re(\gamma) > 0.$$

Chandel [1,2] introduced and studied generalized Stirling numbers whose generating relation is given by

$$\sum_{n=0}^{\infty} S^{(\alpha\ k)}(n,m,r;\frac{z^n}{n!} = \frac{(-1)^m}{m!}\,[1-(k-1)z]^{-\alpha/(k-1)}$$

$$\left[1-\{1-(k-1)z\}^{-r/(k-1)}\right]^m, \quad k \neq 1.$$

Thus an appeal to the Theorem 1, with $x=1$, $\beta = -m$,
$B(z) = [1-(k-1)z]^{-1/(k-1)}$ and $C(z) = [1-(k-1)z]^{-r/(k-1)}$
gives

$$(4.3)\quad S^{(\alpha+\lambda\gamma,\, k)}(n, m+\mu\gamma, r)$$

$$= \sum_{i=0}^{n} \binom{n}{i} \binom{n+\gamma}{i+\gamma} \frac{(m\ \mu i)![(\mu\ i+\gamma)!]}{(m-\mu\gamma)!}$$

$$S^{(\alpha-\lambda_1,\, k)}(i, m-\mu i.\ r)\ S^{(\lambda(i+\gamma),\, k)}(n-i,\ \mu(i+\gamma),\ r),\ Re(\gamma) > 0$$

For $r=q=1$, $s=0$ the relation (19) due to Srivastava and Buschman [7] reduces to

$$\sum_{n=0}^{\infty} \frac{(-1)^n\,(-\alpha)_n\, z^n}{n!}\ {}_{p+1}F_p\left[\begin{matrix} -n,\ \beta,\ \Delta(p-1,\ -\alpha+n); \\ \Delta(p,\ -\alpha); \end{matrix}\ \frac{(p-1)^{p-1}}{p^p}\,x\right]$$

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$$= (1+z)^{\alpha}\left[1-\frac{xz}{(1+z)^{p}}\right]^{-\beta}.$$

Therefore taking $B(z) = 1+z$, $C(z) = \dfrac{z}{(+z)^{p}}$ and making an appeal to the Theorem 1, we obtain

$$(4.4)\quad (\alpha+\lambda\gamma)_n \; {}_{p+1}F_p\left[\begin{matrix}-n,\ \beta+\mu\gamma & \triangle(p-1, \alpha+\lambda\gamma+n); \\ & \triangle(p,\ \alpha+\lambda\gamma);\end{matrix}\ x\right]$$

$$= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{n+\gamma}{k+\gamma}\right)(\alpha-\lambda k)_k\ (\lambda(k+\gamma)_{n-k}$$

$${}_{p+1}F_p\left[\begin{matrix}-k,\ \beta-\mu k, & \triangle(p\ 1,\ \alpha-\lambda k+k); \\ & \triangle(p\ \alpha-\lambda k);\end{matrix}\ x\right]$$

$${}_{p+1}F_p\left[\begin{matrix}-(n-k),\ \mu' k+\gamma). & \triangle(p-1,\ \lambda(k+\gamma)\ +\ n\ k); \\ & \triangle(p,\ \lambda\ k+\gamma));\end{matrix}\ x\right].$$

Acknowledgments

The authors are thankful, to Professor H. M. Srivastava for his suggestions regarding the presentation of the paper.

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

A PARTICULAR ASYMPTOTIC BEHAVIOUR OF THE ZEROS OF σ- ORTHOGONAL POLYNOMIALS

By

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(Received: March 25, 1992)

ABSTRACT

We study the asymptotic behaviour of the unique solution $(t_1^{(m)},\ldots,t_m^{(m)})$ of the system

$$\int_{-1}^{1} p(x)x^k \prod_{i=1}^{m} (x-t_i)^{2s_i+t}\, dx \qquad (k=0,1,\ldots,m-1),$$

satisfying the condition $-1 < t_1^{(m)} < t_2^{(m)} < \ldots < t_m^{(m)} < 1$ for every fixed $m \in N$, when only one of the integers s_i tends to infinity.

1. INTRODUCTION. In this paper we are concerned with the asymptotic behaviour of the unique solution $(t_1^{(m)},\ldots,t_2^{(m)},\ldots,t_m^{(m)})$ of the system

$$(1.1)\quad \int_{-1}^{1} p(x)\, x^k \prod_{i=1}^{m} (x-t_i)^{2s_i+1}\, dx=0, \quad (k=0,1,\ldots,m-1),$$

satisfying the condition

$$(1.2)\quad -1 < t_1^{(m)} < t_2^{(m)} < \ldots < t_m^{(m)} < 1.$$

The system (1.1) arises when we consider a generalization of Turan quadrature formula i. e. a quadrature formula with nodes having arbitrary multiplicity:

$$(1.3)\quad \int_{-1}^{1} p(x)\, f(x)\, dx = \sum_{i=1}^{m} \sum_{k=0}^{2s_i} A_{ik}\, f^{(k)}(x_i) + R[f]$$

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where $p(x) \in L[-1,1]$, $p(x) > 0$ i. e. in $[-1,1]$, m is an arbitrary positive integer, $s_1, s_2, \ldots, s_m$ are the first m terms of a sequence σ of nonnegative integers.

If we assume $-1 < x_1 < x_2 < \ldots < x_m < 1$ the quadrature formula (1.3) attains tne maximum degree of accuracy $n = 2\left(\sum_{i=1}^{m} s_i + m\right)$ (It means that $R[f] = 0$ when f is an arbitrary polynomial of degree $\leqslant n-1$) if and only if $(x_1, x_2, \ldots, x_m)$ is the unique solution of the system (1.1), i. e. $x_i = t_i^{(m)}$, $(i = 1, 2, \ldots, m)$.

The system (1.1) defines a sequence of polynomials

$P_{\sigma,m}(x) = \prod_{i=1}^{m} (x - t_i^{(m)})$ called σ- orthogonal polynomials (Ghizzetti-Ossicini [2]).

In this paper we are concerned with asymptotic behaviour of the solution of (1.1) when only one of the integers s_i goes to infinity.

The following results are obtained:

(1.4) $m \in N$, $\lim_{s_1 \to \infty} t_1^{(m)} = 0$; (This result was proved in [9])

(1.5) $m \geqslant 2$, $\lim_{s_i \to \infty} t_i^{(m)} = 1$ $(i = 2,3,\ldots,m)$,

$$\lim_{s_m \to \infty} t_m^{(m)} = 0 ,$$

$$\lim_{s_m \to \infty} t_i^{(m)} = -1 \quad (i = 1,2,\ldots,m-1),$$

(1.6) $m \geqslant 3$ $\lim_{s_j \to \infty} t_j^{(m)} = 0$ $(j = 2,3,\ldots,m-1)$

$$\lim_{s_j \to \infty} t_i^{(m)} = -1 \quad (i = 1,2,\ldots, j-1)$$

$$\lim_{s_j \to \infty} t_k^{(m)} = 1 \quad (k = j+1, j+2, \ldots, m).$$

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2. Case, when $s_2 = s_3 = \ldots = s_m = 0$.

In the case $s_2 = s_3 = \ldots = s_m = 0$ the system (1.1) is equivalent to

$$(2.1)\begin{cases} \int_{-1}^{1} p(x)\,(x-t)^{2s} \prod_{i=1}^{m} (x-t_i)\,dx = 0, \\ \int_{-1}^{1} p(x)\,(x-t_1)^{2s} \prod_{i=1}^{m} (x-t_i) \prod_{h=m-k}^{m} (x-t_h)\,dx = 0, \ (k=0, 1, \ldots, m-2), \end{cases}$$

with $s = s_1$,

The equation of (2.1) corresponding to $k=m-3$ (For $m=2$ the first equation of (2.1) must be considered) can be written as

$$(2.2) \quad \int_{-1}^{1} p_m(x,s)\,(x-t_1)^{2s+1}\,(x-t_2)\,dx = 0,$$

where $p_m(x,s) = p(x) \prod_{i=3}^{m} (x-t_i)^2$.

We observe that condition $-1 \leq t_i \leq 1$, $(i=3,4,\ldots,m)$, implies $0 \leqslant p_m(x,s) \leqslant 2^{2(m-2)}\,p(x)$ uniformly for $x \in [-1,1]$.

It follows

$$(2.3) \quad \int_{-1}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+2}\,dx + \left(t_1^{(m)} - t_2^{(m)}\right) \int_{-1}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+1}\,dx = 0$$

which implies $\int_{-1}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+1}\,dx > 0$, because $t_1^{(m)} < t_2^{(m)}$.

Now we state:

Theorem I $\lim_{s\to\infty} t_i^{(m)} = 1$, $i=2,3,\ldots m)$

Proof. Let us first prove $\lim_{s\to\infty} t_2^{(m)} = 1$.

Form (2.3) we have

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$$(2.4)\quad t_2^{(m)} - t_1^{(m)} = \frac{\int_{-1}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+2}\,dx}{\int_{-1}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+1}\,dx} \geqslant \frac{\int_{t_1^{(m)}}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+2}\,dx}{\int_{t_1^{(m)}}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+1}\,dx}$$

Hölder's inequality gives

$$(2\cdot5)\quad \int_{t_1^{(m)}}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+1}\,dx \leqslant \left(\int_{t_1^{(m)}}^{1} p_m(x,s)(x-t_1^{(m)})^{2s+2}\,dx\right)^{(2s+1)/(2s+2)} \cdot \left(\int_{t_1^{(m)}}^{1} p_m(x,s)\,dx\right)^{1/(2s+2)} .$$

Hence

$$(2,6)\quad t_2^{(m)} - t_1^{(m)} \geqslant \frac{\left(\int_{t_1^{(m)}}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+2}\,dx\right)^{1/(2s+2)}}{\left(\int_{t_1^{(m)}}^{1} p_m(x,s)\,dx\right)^{1/(2s+2)}} .$$

It is obvious $\lim_{s\to\infty}\left(\int_{t_1^{(m)}}^{1} p_m(x,s)\,dx\right)^{1/(2s+2)} = 1$.

Now we must show

$$(2.7)\quad \lim_{s\to\infty}\left(\int_{t_1^{(m)}}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+2}\,dx\right)^{1/(2s+2)} = 1 .$$

We remark: i) $\forall\, x \in [\,t_1^{(m)}, 1\,] \Rightarrow x-t_1^{(m)} \leqslant 1-t_1^{(m)}$; ii) since $\lim_{s\to\infty} t_1^{(m)} = 0$, for every $\varepsilon \geqslant 0$ there exists $s_\varepsilon \in N$ such that for every $s \geqslant s_\varepsilon\,(s \in N)$, $|\,t_1^{(m)}| < \frac{\epsilon}{2}$ holds.

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Then, for $s > s_{\epsilon_2}$

$$\left(\int_{t_1^{(m)}}^{1} p_m(x,s)\,(x-t_1^{(m)})^{1/(2s+2)}\,dx\right)^{1/(2s+2)} > \left(\int_{1-\epsilon/2}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+2}\,dx\right)^{1/(2s+2)} \geqslant$$

$$\geqslant (1-\epsilon/2-t_1^{(m)})\left(\int_{1-\epsilon/2}^{1} p_m(x,s)\,dx\right)^{1/(2s+2)} > (1-\epsilon)\left(\int_{1-\epsilon/2}^{1} p_m(x\,s)\,dx\right)^{1/(2s+2)}.$$

Hence

$$(2.8)\quad 1-\epsilon < \lim_{s\to\infty}{}'\,(t_2^{(m)} - t_1^{(m)}) = \lim_{s\to\infty}{}'\,t_2^{(m)} \leqslant \lim_{s\to\infty}{}''\,t_2^{(m)} \leqslant 1,$$

then

$$(2.9)\quad \lim_{s\to\infty} t_2^{(m)} = 1 \quad \forall\ m \in N,\ m \geqslant 2$$

Because of $-1 < t_1^{(m)} < \ldots < t_m^{(m)} < 1$, $\forall\ s \in N$, and (2.9) theorem *I* is proved.

3. General case.

The system (1.1) is equivalent to

$$\begin{cases} \int_{-1}^{1} p(x) \prod_{i=1}^{m} (x-t_i)^{2s_i+1}\,dx = 0 \\ \int_{-1}^{1} p(x) \prod_{i=1}^{m} (x-t_i)^{2s_i+1} \prod_{h=m-k}^{m} (x-t_h)\,dx = 0 \quad (k=0,1,\ldots,m-2). \end{cases}$$

If $m=2$ the first equation of (3.1) has to be considered and

$$P_2^*(x,s_1) = P(x)\,(x-t_2)^{2s_2+2}.$$

Therefore the equation of (3.1) corresponding to $k=m-3$ can be written as

$$\int_{-1}^{1} P_m^*(x,s_1)\,(x-t_1)^{2s_1+1}\,(x-t_2)\,dx = 0,$$

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where $p_m^*(x, s_1) = p(x)\,(x-t_2)^{2s_2} \prod_{i=3}^{m} (x-t_i)^{2s_i+2}$.

We observe that condition $-1 \leqslant t_i < 1$, $(i=2,3,..,m)$, implies

$$0 \leqslant p_m^*(x, s_1) \leqslant 2^H p(x), \text{ where } H = 2\left(\sum_{i=2}^{m} s_i + m - 2\right)$$

uniformly for $x \in [-1,1]$.

We first state

Theorem II $\lim_{s_1 \to \infty} t_i^{(m)} = 1$, $(i=2, 3,...,m)$.

The proof is the same as that of Theorem I.

Now we can prove

Theorem III $\lim t_m^{(m)} = 0$; $\lim t_i^{(m)} = -1$, $(i=1,2,...,m-1)$

Proof: Let us write (1.1) in this way

$$\int_{-1}^{1} p(-x)\, x^k \prod_{i=1}^{m} (x + t_i)^{2s_i+1}\, dx = 0, \ (k=0,1,..,m-1).$$

This system has unique solution

$(-t_m^{(m)}, -t_{m-1}^{(m)}, ..., -t_1^{(m)})$ with

$-1 \leqslant -t_m^{(m)} < -t_{m-1}^{(m)} < ... < -t_1^{(m)} < 1$, which verifies,

because of Theorem II, $\lim(-t_m^{(m)}) = 0$, $\lim(-t_i^{(m)}) = 1$, $(i=1,2,..,m-1)$, then Theorem III is proved.

Let us consider $m \geqslant 3$. When $s_j \to \infty$, $(2 \leqslant j \leqslant m-1)$, and s_i are fixed $(i=1,2,..,m;\ i \neq j)$, we state

Theorem IV $\lim_{s_j \to \infty} t_i^{(m)} = 0$.

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Proof. We shall show

$$\lim_{s_j \to \infty}{}' \, t_j^{(m)} = \lim_{s_j \to \infty}{}'' \, t_j^{(m)} = 0, \quad 2 \leqslant j \leqslant m-1.$$

If $\lim_{s_j \to \infty}{}'' \, t_j^{(m)} = T_j > 0$, there would be a subsequence $\{ t_{j,\nu_h}^{(m)} \}$ converging to T_j, then $t_{j,\nu_h}^{(m)} > 0$ definitely.

The last equation of (3.1) can be replaced by

$$(3.2) \quad \int_{-1}^{1} \Big(p(x) \prod_{\substack{i=1 \\ i \neq j}}^{m} (x-t_i)^{2s_i+2} (x-t_j)^{2s_j+1} \, dx = 0 \, .$$

Consider the sequence $(t_{1,\nu_h}^{(m)}, t_{2,\nu_h}^{(m)}, \ldots, t_{m,\nu_h}^{(m)})$, $(h=1,2..)$; let

$$\bar{P}_m(x,s_j) = p(x) \prod_{\substack{i=1 \\ i \neq j}}^{m} (x-t_{i,\nu_h}^{(m)})^{2s_i+2},$$

$0 \leqslant \bar{P}_m(x,s_j) \leqslant 2^L p(x)$ where $L = 2 \big(\sum_{i=1}^{m} s_i + m-1 \big)$,

$\forall \; x \in [-1,1]$ uniformly.

From (3 2) we have

$$0 = \int_{-1}^{-1 + \frac{1}{2} t_{j,\nu_h}^{(m)}} \bar{P}_m(x,s_j) \, (x-t_{j,\nu_h}^{(m)})^{2\nu_h+1} \, dx + \int_{-1 + \frac{1}{2} t_{j,\nu_h}^{(m)}}^{t_{j,\nu_h}^{(m)}} \bar{P}_m(x,s_j) \, (x-t_{j,\nu_h}^{(m)})^{2\nu_h+1} \, dx +$$

$$+ \int_{t_{j,\nu_h}^{(m)}}^{1} \bar{P}_m(x,s_j) \, (x-t_{j,\nu_h}^{(m)})^{2\nu_h+1} \, dx \, .$$

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When $h \to \infty$, the first integral $\to -\infty$, the second one is negative, the third one is bounded. Hence $T_j \leqslant 0$.

By the same method $lim'\, t_j^{(m)} \geqslant 0$ can be proved; then theorem IV is proved.

Finally we can prove

Theorem V (i) $\lim\limits_{s_j \to \infty} t_i^{(m)} = -1$, $(i = 1,2,\ldots,j-1)$;

(ii) $\lim\limits_{s_j \to \infty} t_k^{(m)} = 1$, $(k = j+1, j+2,\ldots,m)$.

Proof. The system (1.1) can be written

$$\begin{cases} \int_{-1}^{1} p(x) \prod_{i=1}^{m} (x-t_i)^{2s_i+1}\, dx = 0 \\ \int_{-1}^{1} p(x) \prod_{i=1}^{m} (x-t_i)^{2s_i+1} \prod_{h=1}^{k} (x-t_h)\, dx = 0 \quad (k = 1,2,\ldots, j-1), \\ \int_{-1}^{1} p(x) \prod_{i=1}^{m} (x-t_i)^{2s_i+1} \prod_{h=1}^{j-1} (x-t_h) \prod_{i=m-r}^{m} (x-t_i)\, dx = 0 \end{cases}$$

$$(r = 0,1,\ldots,m-j-1).$$

Let us write the equation corresponding to $r = m-j-2$ in the form:

$$\int_{-1}^{1} P_m(x,s_j)(x-t_j)^{2s_j+1}(x-t_{j+1})\, dx = 0,$$

where $P_m(x,s_j) = \dfrac{p(x)}{(x-t_{j+1})^2} \prod\limits_{\substack{i=1 \\ i \neq j}}^{m} (x-t_i)^{2s_i+2}$

Because of Theorem II $\lim\limits_{s_j \to \infty} t_{j+1}^{(m)} = 1$, hence (ii) follows from (1.2).

By the same method and because of Theorem III, (i) can be proved.

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Jñānābhə, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

INTEGRATION OF CERTAIN H-FUNCTIONS OF SEVERAL VARIABLES WITH RESPECT TO THEIR PARAMETERS

By

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(Received: November 5, 1991)

ABSTRACT

The object of the present paper is to obtain some interesting results by integrating the multivariable H-function with respect to its parameters. Such integrals are useful in the study of certain boundary value problems.

1. INTRODUCTION

The H-function of several complex variables represented by means of the multiple contour integral was defined by H. M. Srivastava and R. Panda in a series of papers ([2] to [6]). We represent it as follows (see also Srivastava et al. [7, p. 251])

$$(1.1)\quad H[z_1,\ldots,z_r] = H^{0\ n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1\ q_1\ ;\ldots;\ p_r,q_1}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\right|$$

$$\left.\begin{matrix} (a_j:\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,p}:(C'_j,\gamma'_j)_{1,p_1};\ldots;(C^{(r)}_j,\gamma^{(r)}_j)_{1,p_r} \\ (b_j:\beta'_j,\ldots,\beta^{(r)}_j)_{1,q}:(d'_j,\delta'_j)_{1,q_1};\ldots;(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r} \end{matrix}\right]$$

$$= \frac{1}{(2\pi w)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)\ldots\phi_r(\xi_r)\,\psi(\xi_1,\ldots,\xi_r)\,z_1^{\xi_1}\ldots z_r^{\xi_r}\,d\xi_1\ldots d\xi_r.$$

where $w = \sqrt{-1}$

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$$(1.2)\quad \phi_i(\xi_i) = \frac{\prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - \delta_j^{(i)}\xi_i) \prod_{j=1}^{n_i} \Gamma(1-c_j^{(i)} + \gamma_j^{(i)}\xi_i)}{\prod_{j=m_i+1}^{q_i} \Gamma(1-d_j^{(i)} + \delta_j^{(i)}\xi_i) \prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)}\xi_i)}$$

$$(\forall\ i = 1, \ldots, r)$$

$$(1.3)\quad \psi(\xi_1, \ldots, \xi_r) = \frac{\prod_{i=1}^{n} \Gamma(1-a_j + \sum_{i=1}^{r} \alpha_j^{(i)}\xi_i)}{\prod_{j=n+1}^{p} \Gamma(a_j - \sum_{i=1}^{r} \alpha_j^{(i)}\xi_i) \prod_{j=1}^{q} \Gamma(1-b_j + \sum_{i=1}^{r} \beta_j^{(i)}\xi_i)}$$

Conditions of convergence etc. can be seen in the papers of Srivastava and Panda referred to above. These conditions will be assumed to hold good in the multivariable H-function occurring in this paper.

As the multivarlable H function is the most general special function and most of the functions occurring in pure and applied mathematics are partcular cases of it, a number of results, known or new, can be derived easily from our results.

2. EVALUATION OF THE INTEGRALS

$$(2.1)\quad I = \frac{1}{2\pi i} \int_{-w\infty}^{+w\infty} \Gamma(a+x)\, \Gamma(b-x)\, \Gamma(c-x)\, e^{\pm w\pi x}$$

$$H^{0,\ n:\ m_1,n_1;\ \ldots;\ m_r,n_r}_{p,q+1:\ p_1,q_1;\ \ldots;\ p_r,q_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} a_j : \alpha_j', \ldots, \alpha_j^{(r)} \end{matrix} \right)_{1,p}$$

$$\left. \begin{matrix} b_j : \beta_j', \ldots, \beta_j^{(r)} \end{matrix} \right)_{1,q}$$

$$(1-d+x:\ h_1, \ldots, h_r) \quad \begin{matrix} : (c_j', \gamma_j')_{1,p_1}; \ldots; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ : (d_j', \delta_j')_{1,q_1}; \ldots; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \Bigg] dx$$

$$= \Gamma(a+b)\, \Gamma(a+c)\, exp(\pm a\pi)\, H^{0,\ n+1}_{p+1,\ q+2:\ p_1,q_1;\ \ldots;\ p_r,q_r} \quad : m_1,n_1;\ldots;m_r\ n_r$$

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$$\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix} (1+a+b+c-d:h_1,\ldots,h_r)\,,\ (a_j:a'_j,\ldots,\ a_j^{(r)})_{1,p}: \\ (1+b-d:h_1,\ldots,h_r),\ (1+c-d:h_1,\ldots,h_r),\ (b_j,\beta'_j,\ldots,\beta_j^{(r)})_{1,q}: \end{matrix}\right.$$

$$\left.\begin{matrix} (c'_j,\gamma'_j)_{1,p_1};\ldots;(C_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (d'_j,\delta'_j)_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{matrix}\right]$$

provided that (1) $h_i > 0$ $(i=1,2,\ldots,r)$

$$(2)\ Re\left[d-a-b-c+h_i\, d_j^{(i)}/\delta_j^{(i)}\right] > 0\,,$$

$$i = 1,2,\ldots,r\,;\quad j = 1,2,\ldots,m_i$$

Proof: In the integrand of (2.1) we replace the H-function of several variables by (1.1); change the order of integration which is justified under the conditions stated above, we get

$$= \frac{1}{(2\pi w)^r}\int_{L_1}\ldots\int_{L_r}\phi_1(\xi_1)\ldots\phi_r(\xi_r)\,\psi(\xi_1,\ldots,\xi_r\ z_1^{\xi_1}\ldots z_r^{\xi_r}\ d\xi_1\ldots d\xi_r$$

$$\frac{1}{2\pi w}\int_{-w\infty}^{+w\infty}\frac{\Gamma(a+x)\,\Gamma(c-x)\,\Gamma(b-x)}{(d\ x\ +\ h_1\xi_1\ \ldots\ +h_r\xi_r)}\ e^{\mp w\pi x}\ dx.$$

Now we evaluate the inner integral using a known integral formula [8, p. 289], noting that it is a hypergeometric function with unit argument. On expressing the resulting expression with the help of (1.1), we obtain (2.1).

Proceeding similarly we easily evaluate the following integrals.

$$(2.2)\quad \frac{1}{2\pi w}\int_{-w\infty}^{+w\infty}\Gamma(a+x)\,\Gamma(b\ x;\ exp\,(\mp\ w\pi x)$$

$$H^{0,\,n+1:\,m_1,n_1;\ldots;\,m_r,n_r}_{p+1,q+1:\,p+1,q1;\ldots;\,p_r,\,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix} (1-c+x:h_1,\ldots,h_r), \\ (1-d+x:k_1,\ldots,k_r), \end{matrix}\right.$$

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$$\left.\begin{matrix}(a_j:\alpha'_j,\cdots,\alpha_j^{(r)})\ 1,p & :(c'_j,\gamma'_j)\ 1,p_1 & ;\ldots; & (c_j^{(r)},\gamma_j^{(r)})\ 1,p_r \\ (b_j:\beta'_j,\cdots,\beta_j^{(r)})\ 1,q & :(d'_j,\delta'_j)\ 1,q_1 & ;\ldots; & (d_j^{(r)},\delta_j^{(r)})\ 1,q_r\end{matrix}\right] dx$$

$$= \Gamma(a+b)\ exp\ (\pm wa\pi)\ H^{0,\,n+2\ :\ m_1,\,n_1;\ \ldots;m_r,n_r}_{p+2,\,q+2:\,p_1,\,q_1;\ \ldots;\ p_r,q_r}$$

$$\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\right| \begin{matrix}(1-a-c:\ h_1,\ldots,h_r),\ (1+a+b+c-d:\ k_1-h_1,\ldots,k_r-h_r),\\ (1+b-d:\ k_1,\ldots,\ k_r),\ (1+c-d:\ k_1-h_1;\ldots;k_r-h_r),\end{matrix}$$

$$\left.\begin{matrix}(a_j:\alpha'_j,\ldots,\alpha_j^{(r)})\ 1,p & :c'_j,\gamma'_j)\ 1\ p_1 & \ldots, & (c_j^{(r)},\gamma_j^{(r)})\ 1,p_r \\ b_j:\ j',\ldots,j^{(r)})\ 1,q & :d'_j,\delta'_j)\ 1,q_1 & \ldots, & (d_j^{(r)},\delta_j^{(r)})\ 1,q_r\end{matrix}\right]$$

provided that

1. $k_i > h_i > 0$ for $i = 1,2,\ldots,r$

2. $Re\left[\ d-a-b-c\ +\ (k_i-h_i)\ d_j^{(i)}/\delta_j^{(i)}\ \right] > 0$

for $i = 1,2,\ldots,r;\quad j = 1,2,\ldots,m_i$

$$(2.3)\quad \frac{1}{2\pi w}\int_{-w\infty}^{+w\infty} \frac{\Gamma(a+x)\ \Gamma(b-x)}{\Gamma(c-x)}\ exp\ (\mp\ w\pi x)\ H^{0,n+1:m_1,n_1,\ldots,m_r,n_r}_{p+1,q:\,p_1,q_1,\ldots,p_r,q_r}$$

$$\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\right| \begin{matrix}(1-d+x:\ h_1,\ldots,h_r),\ (a_j:\ \alpha'_j,\ldots,\alpha_j^{(r)})\ 1,p\\ (b_j:\ \beta'_j,\ldots,\beta_j^{(r)})\ 1,q\end{matrix}$$

$$\left.\begin{matrix}:(c'_j,\gamma'_j)\ 1,p_1 & ,\ldots, & (c_j^{(r)},\gamma_j^{(r)})\ 1,p_r \\ :(d'_j,\delta'_j)\ 1,q_1 & ,\ldots, & (d_j^{(r)},\delta_j^{(r)})\ 1,q_r\end{matrix}\right] dx$$

$$= \frac{\Gamma(a+b)\ \Gamma\ c-a-b-d)}{\Gamma(c-b)}\ exp\ (\pm\ w\pi a)\ H^{0,\,n+1:\,m_1\,n_1,\ldots,m_r,n_r}_{p+2,q+1:\,p_1,q_1,\ldots,\,p_r,\,q_r}$$

$$\left[\begin{matrix}(-z_1)^{h_1}\\ \vdots\\ (-z_r)^{h_r}\end{matrix}\right| \begin{matrix}(1-a-d:h_1,\ldots,h_r),\ (a_j,\ \alpha'_j,\ldots,\alpha_j^{(r)})\ 1,p,\\ (b_j,\ \beta'_j,\ldots,\beta_j^{(r)})\ 1,q,\end{matrix}$$

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$$\left.\begin{array}{ll}(c\text{-}d{:}h_1,\ldots,h_r) & : (c'_j, \gamma'_j)_{1,\,p_1};\ldots,(c_j^{(r)}, \gamma_j^{(r)})_{1,\,p_r} \\ (c\text{-}a\text{-}b\text{-}d{:}h_1,\ldots,h_r) & : (d'_j, \delta'_j)_{1,\,q_1},\ldots;(d_j^{(r)}, \delta_j^{(r)})_{1,\,q_r}\end{array}\right]$$

provided that

1. $h_i \triangleright 0$, for $i = 1,2,\ldots,r$

2. $$Re\left[c\text{-}a\text{-}b\text{-}d + \max h_i \left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}}\right)\right] > 0$$

for $i = 1,2,\ldots,r$; $j = 1,2,\ldots,n_i$.

3. PARTICULAR CASES

(i) If we take all greek letters equal to unity in (1.1), the multivariate H-function would reduce to the relatively more familiar G-function of two variables. Thus our results (2.1), (2.2) and (2.3) will yield similar integrals involving the double G-function.

(ii) If we set $r = 2$, $m_2 = 1$, $q_2 = q_2 + 1$ in (2.1) we get an integral relation for the H-function of two variables in the form:

$$\frac{1}{2\pi w}\int_{-w\infty}^{+w\infty} \Gamma(a+x)\,\Gamma(b-x)\,\Gamma(c-x)\,exp\,(\mp\pi x)\,H_{p,\,q:\,p_1,\,q_1,\,p_2,\,q_2}^{0,n:m_1,n_1,m_2,n_2}$$

$$\left[\begin{array}{l|l} z_1 & (a_j{\cdot}a'_j, \alpha''_j)_{1,\,p} \\ z_2 & (b_j{:}\beta'_j, \beta''_j)_{1,\,q};\ (1+d+x : h_1,h_2);\end{array}\right.$$

$$\left.\begin{array}{l}(c'_j, \gamma'_j)_{1,\,p_1};\ c''_j, \gamma''_j)_{1,\,p_2} \\ (d'_j, \delta'_j)_{1,\,q_1};\ (d''_j, \delta''_j)_{1,\,q_2}\end{array}\right] dx$$

$$= \Gamma(a+b)\,\Gamma(a+c)\,exp\,(\pm a\pi)\,H_{p+1,q+2\,:\,p_1,q_1;p_2,q_2+1}^{0,\,n+1\,:\,m_1,n_1;1,n_2}$$

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$$\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\right| \begin{matrix} (1+a+b+c-d : h_1, h_2) \\ (1+b-d : h_1, h_2), \end{matrix} \quad \begin{matrix} \\ (1+c-d : h_1, h_2), \end{matrix}$$

$$\left.\begin{matrix} (a_j : \alpha'_j, \alpha''_j)_{1,p} : (c'_j, \gamma'_j)_{1,p_1} ; (c''_j, \gamma''_j)_{1,p_2} \\ (b_j : \beta'_j, \beta''_j)_{1,q} : (d'_j, \delta'_j)_{1,q_1} ; (d''_j, \delta''_j)_{1,q_2} \end{matrix}\right] \quad \ldots (3.1)$$

with the conditions

1. $h_1 \geqslant 0, h_2 \geqslant 0$

2. $Re\left[d-a-b-c+h_1 \sum_{j=1}^{m_1} \frac{d'_j}{\delta'_j} + h_2 \sum_{j=1}^{m_2} \frac{d''_j}{\delta''_j}\right] > 0$

(iii) In (3.1), if we let $p=q=n=0, n_2 = p_2 = 0, m = 1 = q_2$, $d''_1 = 0$, $\delta''_1 = 0$ and take limit $z_2 \to 0$; and arranging the parameters we get an integral relation proved earliear by Nair and Nambudripad [1].

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

ON A DISTRIBUTIONAL MEIJER-BESSEL TRANSFORM

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(Received: February 9, 1992)

ABSTRACT

In this paper two new characterizations of a generalized Meijer Transform for distributions have been developed, using two transfrmations on the space of distribution viz the dilations U_n and the exponential shifts T^{-p}. The standard theorems on analyticity, uniqueness, invertibility and some standard operation transform formulas are proved, using the new characterization as the definition of the generalized Meijer Transform,

1. INTRODUCTION

A grneralization of Meijer transformation called Meijer-Bessel transform $F_{(s)}$ of the generalized function $f \in M_{\mu,a}$ (I), $I = (0,\infty)$ and defined as

$$F(s) = (2/\pi)^{\frac{1}{2}} \int_0^\infty (st)^{y+\frac{1}{2}} K_\mu (st, f(t)\, dt$$

or, $F.s) = < f(t), (2/\pi)^{\frac{1}{2}} (st)^{\lambda+\frac{1}{2}} K_\mu (st) >$ (1.1)

have been studied by Agrawal [1] where $K_\mu(z)$ denotes the Bessel function of third kind and $K_\mu(z)$ has the integral representation

$$K_\mu(z) = \frac{\sqrt{\pi}}{\Gamma(\mu+\frac{1}{2})} \left(\frac{z}{2}\right)^\mu \int_{-1}^\infty (t^2-1)^{\mu-\frac{1}{2}}\ e^{-zt}\, dt$$

for $Re\ \mu > -\frac{1}{2}$, $Re\ z > 0$ [Erdèlyi, 5, p. 18]

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Also $M_{\mu,a}(I)$ is the dual of the testing function space $M_{\mu,a}(I)$ defined in this paper.

In this paper we give two new characterizations of the Meijer-Bessel transform for distribution with the help of the dilations U_n and the exponential shifts T^{-p} introduced earlier by Gesztelyi [6]. It is interesting to note here that Gesztelyi considered two transformations viz., dilations U_n and exponential shifts T^{-p} which are defined for ordinary function f, complex number p, and positive integer n by

$$U_n f(t) = n f(nt) \tag{1.3}$$

$$T^{-p} f(t) = e^{-pt} (f)(t) \tag{1.4}$$

Gesztelyi shows that whenever the sequence $\{U_n f\}$ converges (in the sense of Mikusinski csnvergence [9] the limit is necessarily a complex number. Also he proved that if f is a function which has Laplace transform at p, then the sequence of functions $\{U_n T^{-p} f(t)\}$ converges (in the Mikusinski sense) as $n \to \infty$ to the classical Laplace transform of f at p. He then defined the Laplace Transform of a Mikusinski operator x as the limit (whenever it exists in the sense of Mikusinski convergence) of the sequence $\{U_n T^{-p} x\}$ and shows that his definition generalizes the previous formulations of the Laplace transform Mikusinski operators of G-Doetsch [4] and V. A. Ditkin ([2] - [3]). Price [10] working on the same time introduced the Laplace transform of a distribution f using the sequence of the form $\{U_n T^{-p} f\}$ and showed that the new definition is equivalent to the Schwortz's extension of the transform to distributions He also introduced spaces B and B_0 and their duals B' and B'_0 has and shows that each distribution f in B'_0 has a unique extension $\overset{v}{f}$ in B'. He also shows that the sequence $\{U_i f\}$ converges $< f, 1 > \delta$, whenever f is in B'_0. Recently the present author ([7] - [8] has given new characterizations for a generalized Laplace and Hermite transform for distributions.

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In fact, B_0 is a subspace of $B(R^n)$ (or where R^n is understood, by B, space of all complex valued functions of an n-dimensional real variable $t = (t_1, t_2, \ldots, t_n)$ which possess contineous and bounded partial derivatives of all orders) consisting of those functions in B each of whose derivatives approaches to zero as $|t| \to \infty$, B'_0 (the dual of B_0) is a subspace of D' and a distribution f in B'_0 is completely determined by its value in D.

2. The Testing Function Spaces $M_{\mu,a}(I)$ and its dual $M'_{\mu,a}(I)$

The testing function space $W_{\alpha,\beta}(I)$, consists of infinitely differentiable complex valued function $\phi(t)$ defined on $I\,(0,\infty)$ such that

$$\rho_{a,k}^{\mu,\lambda}(\phi) = \sup_{0<t<\infty} \left| e^{at}\, t^{\mu-\lambda-\frac{1}{2}}\, S^{k}_{\mu\,\lambda\,t}\, \phi\, t) \right|$$

is finite, where

$$S_{\mu,\lambda,t} \equiv t^{\lambda-\mu-\frac{1}{2}}\, Dt^{2\mu+1}\, Dt^{-\lambda-\mu-\frac{1}{2}}$$

and a is a real number and $Re\ \mu > 0$.

The topography over $M_{\mu,a}(I)$ is generated by the collection of seminorms $\left\{ \rho_{a,k}^{\mu,\lambda} \right\}_{k=0}^{\infty}$ The dual of $M_{\mu,a}(I)$ is denoted by $M'_{\mu,a}(I)$ which consists of all continuous linear functionals on $M_{\mu,a}(I)$.

3. Two New Characterizations of the Meijer-Bessl Transform:

In this section we give two new characterizations of the Meijer-Bessel Transform for one-dimensional distributions

We will say that a distribution f is Meijer-Bessel transformable if there exists an open interval (c,d) such that whenever $p = \frac{1}{t} \log \left[\left(\frac{2}{\pi} \right)^{\frac{1}{2}} (st)^{\lambda+\frac{1}{2}} K_\mu(st) \right]$ ($t \neq 0$ is a complex number) is a complex number with real part in (c,d), $T^{-p} f$ is a distribution in B'_0 where B'_0 is the dual of B_0, a subspace of D' as defined by Price [10].

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If (c,d) is the largest such open interval then the set $\Omega = \{p: Re\ p \in (c,d)\}$ is called the domain of the definition of the Meijer-Bessel traasform of f.

If f is a Meijer-Bessl transformable distribution where the transform has domain of definition Ω, then for $p \in \Omega$, we define the Meijer-Bessel transform $MB[f](p)$ of f at p by

$$MB[f](p) = \frac{1}{\phi(0)} \lim_{j \to \infty} < U, T^{-p} f, \phi > \tag{3.1}$$

where ϕ is a test function in D with $\phi(0) \neq 0$.

We have another characterization also as

$$MB[f](p) = \langle T^{-p} f, 1 \rangle \tag{3.2}$$

where $p = -\frac{1}{t} log\ [(2/\pi)^{\frac{1}{2}} (st)^{\lambda+\frac{1}{2}} K_{\mu}(st)]$.

From (3.2) we see that $MB(f)(p)$ is a complex valued function of the complex variable p with domain Ω. In fact, the mapping MB is linear. For if f and g are distributions that are transformable at p and α, β are complex numbers then $(\alpha f + \beta g)$ is Meijer-Bessel transformable at p and

$$MB[\alpha f + \beta g](p) = < T^{-p}(\alpha f + \beta g), 1 >$$
$$= \{\alpha < T^{-p} f, 1 > + \beta \langle T^{-p} g, 1 \rangle\}$$
$$= \alpha\ MB[f](p) + \beta\ MB[g](p).$$

Theorem 3.1: (Analyticity theorem):

If f is a distribution that is Meijer-Bessel transformable in Ω, then $MB[f](p)$ is analytic function of p in Ω and

$$\frac{d}{dp} MB[f](p) = MB[-t f(t)](p) \tag{3.3}$$

Proof: Let $\Omega = \{p: Re\ p \in (c,d)\}$ and

$$p = -\frac{1}{t} log\ [(2/\pi)^{\frac{1}{2}} (st)^{\lambda+\frac{1}{2}} K_{\mu}(st)];\ t \neq 0,$$

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Let us choose p_0 in Ω and ϵ in (0,1) such that
$\epsilon < min \{Re\ p_0\text{-}c,\ d\text{-}Re\ p_0\}$

If $\lambda(t) = e^{\epsilon t} + e^{-\epsilon t}$ then $\frac{1}{\lambda}$ is in $S \subset B_0$

and $(\lambda\, T^{-p_0} f)$ is in B'_0. Also as long as $|p - p_0| < \epsilon$, we have
$MB[f](p) - MB[f](p_0)$

$$= < \frac{e^{-pt} - e^{-p_0 t}}{p - p_0} f(t),\ 1(t) >$$

$$= < \lambda(t)\, e^{-p_0 t} f(t),\ \frac{1}{\lambda(t)} \left[\frac{e^{-(p-p_0)t}}{p-p_0} \right] >$$

$$= < \lambda(t)\, e^{-p_0 t} f(t),\ -\frac{1}{\lambda(t)} + \frac{(p-p_0)t^2}{\lambda(t)} \sum_{j=2}^{\infty} \frac{[-(p-p_0)t]^{j-2}}{j!} >$$

But each derivative of

$\frac{t^2}{\lambda(t)} \sum_{j=2}^{\infty} \frac{[-(p-p_0)t]^{j-2}}{j!}$ is bounded in absolute value by the corresponding derivative of $\frac{t^2}{\lambda(t)}\ exp.\ |(p-p_0)t|$ and is, therefore, in S. Thus as $p \to p_0$ $\frac{1}{\lambda(t)} \left[\frac{e^{-(p-p_0)t} - 1}{p - p_0} \right]$ converges in B_0 to $-\frac{t}{\lambda(t)}$ and we have

$$\frac{d}{dp_0} MB[f](p_0) = \lim_{p \to p_0} \frac{MB[f](p) - MB[f](p_0)}{p - p_0}$$

$$= < \lambda(t) T^{-p_0} f(t),\ -\frac{t}{\lambda(t)} >$$

$$= < T^{-p_0} [-t f(t)],\ 1(t) >$$

$$= MB[t f(t)](p_0)$$

4. Treatment of the Convolution of Two Distributions:

Much of the usefulness of Meijer-Bessel transform is a result of

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the way it treats the convolution of two distributions. We give here this important property of the transformation by the following theorem.

Theorem 4.1 : If f and g are Meijer-Bessel transformable distributions such that the domain of their tespective transforms have intersection Ω, then $f*g$ is Meijer-Bessel transformable in Ω and for every $p\epsilon\Omega$, we have.

$$MB[f*g](p) = MB[f](p)\, MB[g](p).$$

Proof: For $p\epsilon\Omega$, $T^{-p}f$ and $T^{-p}g$ are both in B'_0. So by Lemma 2.3 (p. 20, Price [10] whica states that if f and g are in $B'_0(R^n)$, then their convolution can be defined also in $B'_0(R_n)$ we have

$$T^{-p}f * T^{-p}g = T^{-p}(f+g) \text{ is in } B'_0.$$

Therefore, $(f_* g)$ is Meijer-Bessel transformable at p and from (3.2) and the definition of convolution, we get

$$\begin{aligned} MB[f*g](p) &= < T^{-p}(f*g),\ 1 > \\ &= < T^{-p}f * T^{-p}g,\ 1 \triangleright \\ &= < T^{-p}f(t) \otimes T^{-p}g(\tau),\ 1(t+\tau) > \\ &= < T^{-p}f(t) \otimes T^{-p}g(\tau),\ 1(t)\ 1(\tau) > \\ &= < T^{-p}f,\ 1 > < T^{-p}g,\ 1 > \\ &= MB[f](p)\ MB[g](p). \end{aligned}$$

5. Inversion and Uniqueness Theorems for the Meijer-Bessel Transformation:

No theory of the Meijer-Bessel transform woud be useful without the inversion and uniqueness theorems. We give Thorem 5.1 which includes both inversion and uniqueness theorems as its corollary.

In what follows we will have as independent variable at various times the real variable t and, real and imaginary parts of complex variable p,

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$$p = - \frac{1}{t} \log [(2/\pi)^{\frac{1}{2}} (st)^{\lambda+\frac{1}{2}} K_{\mu}(st)]; \ 0 < s < \infty,$$

$t \neq 0$. For this reason we will some time indicate particular independent variable for a space or an operation by a subscript ξ

e. g. $< f(\xi), e^{-i\omega\xi} >_{\xi}$, where $f(\xi)$ is in B'_0 and ω is a parameter.

Theorem 5.1: If f is a distribution in B'_0 then

$$f(t) = \frac{1}{2\pi} \lim_{r\to\infty} \int_{-r}^{r} e^{i\omega t} < f(\xi), e^{-i\omega\xi} >_{\xi} d\omega \qquad (5.1)$$

where the limit is taken in D'_t.

Proof: The proof is the same as that of Laplace transform as given in [10].

There are three corollaries of this theorem.

Corollary 5.1 : (a) If σ is a real number such that $e^{-\sigma t} f(t)$ is in $B'_{0\xi}$, then as distributions,

$$f(t) = \lim_{r\to\infty} \frac{1}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} e^{pt} < e^{-p\xi} f(\xi), 1(\xi) >_{\xi} dp.$$

Proof: If $e^{-\sigma t} f(t) \in B'_{0\xi}$, then as long as $Re.\ p = \sigma$, $e^{-pt} f(t) \in B'_{0\xi}$, and

$$e^{-\sigma t} f(t) = \frac{1}{2\pi} \lim_{r\to\infty} \int_{-r}^{r} e^{i\omega t} < e^{-\sigma\xi} f(\xi), e^{-i\omega\xi} >_{\xi} d\omega$$

Thus we have,

$$f(t) = \frac{1}{2\pi} \lim_{r\to\infty} \int_{-r}^{r} e^{\sigma t} e^{i\omega t} < e^{-\sigma\xi} f(\xi), e^{-i\omega\xi} >_{\xi} d\omega$$

$$= \frac{1}{2\pi i} \lim_{r\to\infty} \int_{\sigma-ir}^{\sigma+ir} e^{pt} < e^{-p\xi} f(\xi), 1(\xi) >_{\xi} dp.$$

Corollary 5.2: (Inversion Theorem)

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If f is a Meijer-Bessel transformable in

$\Omega = \{ p: c < Re\, p < d \}$, $p = -\frac{1}{t} \log [(2/\pi)^{\frac{1}{2}} (st)^{\lambda+\frac{1}{2}} K_\mu (st)]$

is a complex number, $t \neq 0$

then as long as $c < \sigma < d$,

$$f(t) = \lim_{r \to \infty} \frac{1}{2\pi i} \int_{\sigma - ir}^{\sigma + ir} e^{pt}\, MB[f](p)\, dp,$$

where the limit is taken in D_t'.

Corollary 5.3 : (Uniqueness Theorem)

If f and g are Meijer-Bessel transformable distributions such that $MB[f](p) = MB(g)(p)$ on some vertical line in the common domain of the transform of f and g then $f = g$ as distributions.

6. Some Standard Operation Transform Formulae for the Distribution Meijer-Bessel Transform :

We will introduce some standard operation transform formulae using the characterizations of the transform given in (3.1) as below.

Let f be a Meijer-Bessel transformable distribution whose transform has domain of definition $\Omega = \{ p: c < Re\, p < d \}$,

where

$$p = -\frac{1}{t} \log [(2/\pi)^{\frac{1}{2}} (st)^{\lambda+\frac{1}{2}} K_\mu(st) f(t)\, dt \quad (s \neq 0)$$

is a complex number. Then $f^{(1)}$ is also Meijer-Bessel transformable in Ω. To compute the transform of $f^{(1)}$, let ϕ be a function in D such that $\phi(0) = 1$, and let $j = 0$ be an integer. Then if $p \in \Omega$, we have

$$< U(T^{-p} f^{(1)}, \phi > = < f^{(1)}(t), e^{-pt} \phi(\frac{t}{j}) >$$

$$= < f(t), pe^{-pt} \phi(\frac{t}{j}) - \frac{1}{j} e^{-pt} \phi^{(1)}(\frac{t}{j}) >$$

$$= p < U_t T^{-p} f, \phi > - \frac{1}{j} < U_t T^{-p} f, \phi^{(1)} >$$

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As $j \to \infty$, the term $\frac{1}{j} < U_j\, T^{-p} f,\, \phi^{(1)} >$ converges to zero and so from (3.1) we have

$$MB[\, f^{(1)}\,](p) = \lim_{j\to\infty} p < U_j\, T^{-p} f,\, \phi > = p\, MB[\, f\,](p) \qquad (6.1)$$

By induction we can prove that for every positive n,

$$MB\,[\, f^{(n)}\,(p) = p\; MB\;[\, f^{(n-1)}]\,(p) = p^n MB[\, f\,]\,(p) \qquad (6.2)$$

Another operation transform formula can be obtained from (3.3) viz.

$$MB\,[\, -t\, f(t)\,]\,(p) = \frac{d}{dp}\; MB\,[\, f\,]\,(p).$$

This formula can be extended by the method of induction for every positive integer n, to get

$$MB\,[t^n f(t)]\,(p) = (-1)^n \frac{d^n}{dp^n}\; MB[\, f\,]\,(p) \qquad (6.3)$$

If f is Meijer–Bessel transformable in Ω, then $f(t-\eta)$ is Meijer–Bessel transformable in Ω for every real number η and we have

$$< U_j\, T^{-p} f\,(t-\eta),\, \phi(t) > = < f(t-\eta),\, e^{-pt}\, \phi\,(\frac{t}{j}) >$$

$$= < f\,(t,\, e^{-p(t+\eta)}\, \phi(\frac{t+\eta}{j}) >$$

$$= e^{-p\eta} < U_j\, T^{-p} f\,(t),\, \phi\,(t+\eta) >$$

Non, $\phi\,(t+\eta) \in D$ and as 1 on g as $\phi(\eta) \neq 0$,

$$\lim_{j\to\infty} e^{-p\eta} < U_j T^{-p} f\,(t),\, \phi(t+\eta) > =$$

$$\frac{1}{\phi(\eta)}\, e^{-p\eta} < T^{-p} f, 1 > < \delta(t),\, \phi(t+\eta) >,$$

So we have $MB\,[\, f\,(t-\eta)]\,(p) = e^{-p\eta} MB\,[\, f\,]\,(p)$.

If q is a fixed complex number and f is Meijer–Bessel transformable in Ω then $e^{-qt} f(t)$ is Meijer–Bessel transformable in $\Omega' = \{\, p:\, c - Re\, q < Re\, p < d - Re\, q \,\}$.

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and we have

$$< U_t\, T^{-p}\,[e^{-qt} f(t)],\ \phi(t) >$$
$$= < U_t\, T^{-(p+q)} f,\ \phi >$$

Thus, whenever $p \in \Omega'$,

$$MB\,[e^{-qt} f(t)]\,(p) = MB\,[f]\,(p).$$

If k is a fixed positive integer and f is Meijer-Bessel transformable in Ω, then $(U_k f)$ is Meijer-Bessel transformable in $\Omega'' = \{p: kc \leqslant Re\ p \leqslant kd\}$. For $p \in \Omega''$ we have

$$\leqslant U_t\, T^{-p}\,\{U_k f\},\ \phi > = \leqslant U_k f,\ e^{-pt}\phi\,(\frac{t}{j}) >$$
$$= < f(t),\ e^{-pt/k}\,\phi\,(t/jk) >$$
$$= \leqslant U_t\, T^{-p/k} f,\ \phi\,(t/k) >$$

As $j \to \infty$, this converges to

$$\leqslant T^{-p/k} f,\ 1 > < \delta,\ \phi >.$$

So, we get the formula.

$$MB\,[U_k f]\,(p) = MB\,[f]\quad p/k\ ).$$

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

VIBRATION ANALYSIS OF A NON-UNIFORM CIRCULAR PLATE SUBJECTED TO ONE-DIMENSIONAL STEADY-STATE TEMPERATURE DISTRIBUTION

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(Received: January 20, 1990; Revised: August 18, 1991)

ABSTRACT

The effect of temperature on the vibration of solid bodies is significant, especially in mechanical constructions, where certain parts of the body have to operate under elevated temperatures. The interest in this direction has highly increased because of rapid developments in space technology, high speed atmospheric flights, and nuclear energy applications The present paper aims at discussing the effect of linearly transient temperature fields on the natural frequencies of a circular plate. The frequencies, deflections and moments corresponding to the first two modes of vibration are computed for clamped and simply supported edge conditions by using Frobenius method.

1. INTRODUCTION

With the advancements of space technology and high sqeed atmospheric flights, it has become necessary to obtain a greate insight

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into the dynamic behaviour of plates working under temperature fields. The thermal dependence of frequency of plates of different shapes is of great importance. The effect of temperature on the modulus of elasticity of materials is far from negligible especially in the design of aircrafts and rockets in which certain parts have to operate under elevated temperatures. An up-to-date account of problems involved in the vibrations of plates has been given by Leissa ([5], [6]). The frequencies and normal modes of a circular plate of a circular plate of parabolically varying thickness with free edge have evaluated by Harris [2]. Jian [3] investigated the free axisymmetric vibrations of clamped and simply supported circular plates of lineally varying thickness. Gupta and Lal [1] have observed the buckling and vibrations of circular plate of variable thickness. The axisymmetric vibrations of circular plates of linearly varying thickness have also been studied by Tomar et al. ([11] to [14]) by taking elastic foundation, shear deformation and nonhomogeneiy into account. Tomar and Tiwari [15] have adopted the linea variation of temperature for the computation of free vibrations of a circular plate of variable thickness by a method based on Rayleigh's quotient. Recently Tomar and Gupta ([8], [9]) considered the effect of harmonic and linear temperature variations of axisymmetric vibrations of orthotropic circular plates of variable thickness respectively.

An attempt is made here to study the effect of linearly varying temperature on the free vibrations of circular plates of linearly varying thickness.

2. EQUATION OF MOTION

For an isotropic circular plate of variable thickness, as derived after expressing the equation of motion [10] in polar coordinates and assuming that the axis of the plate coincides with the radial direction,

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and that the thickness of the plate h, Young's modulue E, density ρ and flexural rigidity D are functions of r, the differential equation is governed by

$$Eh^2\frac{\partial^4 w}{\partial r^4} + 2\left[h^2\frac{\partial E}{\partial r} + \left\{3h\left(\frac{\partial h}{\partial r}\right) + \frac{h^2}{r}\right\}E\right]\frac{\partial^3 w}{\partial r^3} + \left[h^2 + \left\{6h\left(\frac{\partial h}{\partial r}\right) + (2+\nu)\frac{h^2}{r}\right\}\frac{\partial E}{\partial r} + \left\{3h\left(\frac{\partial^2 h}{\partial r^2}\right) + 6\left(\frac{\partial h}{\partial r}\right)^2 - \frac{h^2}{r^2} + (2+\nu)\frac{3h}{r}\left(\frac{\partial h}{\partial r}\right)\right]E\right]\frac{\partial^2 w}{\partial r^2} + \left[\frac{\nu h^2}{r}\frac{\partial^2 E}{\partial r^2} + \left\{\frac{6\nu h}{r}\left(\frac{\partial h}{\partial r}\right) - \frac{h_2}{r^2}\right\}\frac{\partial E}{\partial r} + \left\{3\nu\frac{h}{r}\left(\frac{\partial^2 h}{\partial r^2}\right) + \frac{6\nu}{r}\left(\frac{\partial h}{\partial r}\right)^2 - \frac{3h}{r^2}\left(\frac{\partial h}{\partial r}\right) + \frac{h^2}{r^3}\right\}E\right]\frac{\partial w}{\partial r} + 12(1-\nu^2)\rho\frac{\partial^2 w}{\partial t^2} = 0 \qquad (1)$$

where w is the transverse displacement and ν the Poisson's ratio.

Introducing the non-dimensional variables $H = \frac{h}{a}$, $R = \frac{r}{a}$ and $W = \frac{w}{a}$, equation (1) reduces to

$$\bar{E}H^2\frac{\partial^4 W}{\partial R^4} + 2\left[H^2\frac{\partial \bar{E}}{\partial R} + \left\{3H\left(\frac{\partial H}{\partial R} + \frac{H^2}{R}\right)\bar{E}\right\}\right]\frac{\partial^3 W}{\partial R^3} + H^2\frac{\partial^2 \bar{E}}{\partial R^2} + \left\{6H\left(\frac{\partial H}{\partial R}\right) + (2+\nu)\frac{H^2}{R}\right\}\frac{\partial \bar{E}}{\partial R} + \left\{3H\left(\frac{\partial^2 H}{\partial R^2}\right) + 6\left(\frac{\partial H}{\partial R}\right)^2 - \frac{H^2}{R^2} + (2-\nu)\frac{3H}{R}\left(\frac{\partial H}{\partial R}\right)\right\}\bar{E}\left]\frac{\partial^2 W}{\partial R^2} + \left[\frac{\nu H^2}{R}\frac{\partial^2 \bar{E}}{\partial R^2} + \left\{6\frac{\nu H}{R}\left(\frac{\partial H}{\partial R}\right) - \frac{H^2}{R^2}\right\}\frac{\partial \bar{E}}{\partial R} + \left\{3\nu\frac{H}{R}\left(\frac{\partial^2 H}{\partial R^2}\right) + \frac{6\nu}{R}\left(\frac{\partial H}{\partial R}\right)^2 - \frac{3H}{R^2}\left(\frac{\partial H}{\partial R}\right) + \frac{H^2}{R^3}\right\}\bar{E}\right]\frac{\partial W}{\partial R} + 12(1-\nu^2)a^2\frac{\bar{\rho}}{\rho}\frac{\partial^2 W}{\partial t^2} = 0, \qquad (2)$$

where 'a' is the radius of the plate, $\bar{E} = \frac{E}{a}$ and $\bar{\rho} = \frac{\rho}{a}$.

Here, the plate material is assumed to be subjected to a linear temperature distribution in radial direction in the following form

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$$T = T_0 (I-R), \tag{3}$$

where T denotes the temperature excess above the reference temperature at any point at a distance $R = \frac{r}{a}$ from the centre and T_0 denotes the temperature excess above the reference temperature at the end $R = 0$.

Furthermore, most of the engineering materials are found to have a linear relationship between the modulus of elasticity and temperature [7], therefore one can have

$$\overline{E}(T) = \overline{E}_0(1-\xi T), \tag{4}$$

where $\overline{E}_0$ is the modulus of elasticity of the material at the reference temperature and ξ is a constant.

Assuming the temperature at $R = 1$ as the reference temperature, the modulus variation becomes

$$\overline{E}(R) = \overline{E}_0 [1-\eta(1-R], \tag{5}$$

where $\eta = \xi T_0$, $0 \leqslant \eta \leqslant 1$, a parameter known as thermal gradient.

Thus by taking eqation (5) with the following type of variation in the thickness of the plate

$$H = H_0 (1 - \beta R), \tag{6}$$

where $H_0 = H \mid_{R=0}$ and β is a taper constant. Equation (2) becomes

$$\left\{ 1-\eta (1-R) \right\} (1-\beta R)^2 \frac{\partial^4 W}{\partial R^4} + 2\eta \left[(1-\beta R)^2 + \left\{ -3\beta(1-\beta R) + \frac{(1-\beta R)^2}{R} \right\} \left\{ 1-\eta(1-R) \right\} \right] \frac{\partial^3 W}{\partial R^3} + - \left[6\eta\beta(1-\beta R) + (2+\nu)\eta \frac{(1-\beta R)^2}{R} + \left\{ 6\beta^2 - \frac{(1-\beta R)^2}{R^2} - 3\beta(2+\nu) \frac{(1-\beta R)}{R} \right\} \left\{ 1-\eta (1-R) \right\} \right] \frac{\partial^2 W}{\partial R^2}$$

$$+ \left[-6\nu\eta\beta \frac{(1-\beta R)}{R} - \frac{\eta(1-\beta R)^2}{R^2} + \left\{ \frac{6\nu\beta^2}{R} + \frac{3\beta(1-\beta R)}{R^2} \right.$$

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$$+\frac{(1-\beta R)^2}{R^3}\Big\}\Big\{1-\eta(1-R)\Big\}\Big]\frac{\partial W}{\partial R}+\frac{(1-\nu^2)\,a^2\,\bar{\rho}}{\bar{E}_0\,I^*}\frac{\partial^2 W}{\partial t^2}=0, \qquad (7)$$

where $I^*=\dfrac{H^2_0}{12}$.

3. SOLUTION

For harmonic vibrations, the substitution

$$W(R,t)=\bar{W}(R)\,e^{i\omega t}$$

is made tnto equation (7), which reduces to

$$\sum_{\lambda=0}^{\infty} a_\lambda[B_1(\lambda)R^{c+\lambda-1}+B_2(\lambda)R^{c+\lambda-3}+B_3(\lambda)R^{c+\lambda-2}$$

$$+B_4(\lambda)R^{c+\lambda+1}+G_{05}R^{c+\lambda}]=0, \qquad (8)$$

where

$$\bar{W}(R)=\sum_{\lambda=0}^{\infty} a_\lambda R^{c+\lambda},\ a_0\neq 0, \qquad (9)$$

is the assumed series solution for $\bar{W}$ and c is the exponent of singularity,

$$B_1(\lambda)=G_{01}b_\lambda^{(3)}+G_{02}b_\lambda^{(2)}+G_{03}b_\lambda^{(1)}+G_{04}b_\lambda^{(0)},$$

$$B_2(\lambda)=G_{11}b_\lambda^{(3)}+G_{12}b_\lambda^{(2)}+G_{13}b_\lambda^{(1)}+G_{14}b_\lambda^{(0)},$$

$$B_3(\lambda)=G_{21}b_\lambda^{(3)}+G_{22}b_\lambda^{(2)}+G_{23}b_\lambda^{(1)}+G_{24}b_\lambda^{(0)},$$

$$B_4(\lambda)=G_{31}b_\lambda^{(3)}+G_{32}b_\lambda^{(2)}+G_{33}b_\lambda^{(1)}+G_{34}b_\lambda^{(0)},$$

$G_{01}=1-\eta,\quad G_{11}=\eta-2(1-\eta)\beta,\quad G_{21}=-2\eta\beta+(1-\eta)\beta^2,$

$G_{31}=\eta\beta^2,\ G_{02}=2(1-\eta),\ G_{12}=4\eta-10(1-\eta)\beta,\ G_{22}=-14\eta\beta+8(1-\eta)\beta^2,$

$G_{32}=10\eta\beta^2,\ G_{03}=-(1-\eta),\ G_{13}=(1+\nu)\eta-(4+3\nu)(1-\eta)\beta,$

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$G_{23} = -(14+5\nu)\ \eta\beta + (11+3\nu)(1-\eta)\beta^2$, $G_{33} = (19+4\nu)\eta\beta^2$,

$G_{04} = 1-\eta$, $G_{14} = (1-\eta)\beta$, $G_{24} = 3(1-2\nu)\eta\beta - 2(1-3\nu)(1-\eta)\beta^2$,

$G_{34} = -3(1-4\nu)\eta\beta^2$, $G_{05} = -\frac{\Omega^2}{1^*}$, $\Omega^2 = \frac{(1-\nu^2)}{E_0}a^2\omega^2$,

ω is the radian frequency and Ω-frequency parameter,

$b_\lambda^{(0)} = c+\lambda$, $b_\lambda^{(1)} = (c+\lambda)(c+\lambda-1)$, $b_\lambda^{(2)} = (c+\lambda)(c+\lambda-1)(c+\lambda-2)$,

For (9) to be the solution, the coefficients of various powers of R in the expression obtained substituting (9) in (8), must be identically zero. Thus by equating to zero the coefficient of the lowest power of R, yields the indicial equation

$$c^2(c-2)^2 = 0$$

whch gives $c = 0, 0, 2, 2$.

Further, equating to zero the coefficients of the next subsequent powers of R, it is found that $a_1 = 0$, a_2 is indeterminate for $c = 0$. Also, the remaining a_λ's ($\lambda = 3,4,\ldots$) can be written in terms of a_0 and a_2. Hence, assuming

$$a_\lambda = F_\lambda^{(0)}\ a_0 + F_\lambda^{(2)}\ a_2, \tag{10}$$

where

$$F_0^{(0)} = F_1^{(2)} = 1,\ F_1^{(0)} = F_2^{(0)} = F_0^{(2)} = F_1^{(2)} = 0,$$

and rest $F_\lambda^{(i)}$'s ($i=0,2$; $\lambda=3,4,\ldots$) can be calculated from the recurrence relation

$$F_\lambda^{(i)} = -[B_2(\lambda-1)\ F_{\lambda-1}^{(i)} + B_3(\lambda-2)\ F_{\lambda-2}^{(i)} + B_4(\lambda-3)\ F_{\lambda-3}^{(i)} + G_{05}\ F_{\lambda-4}^{(i)}]/B_1(\lambda), \tag{11}$$

with the consideration that

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$$B_j(\lambda-s) = \begin{cases} B_j(\lambda-s) & \text{if } \lambda-s \geqslant 0, \\ 0 & \text{if } \lambda-s < 0. \end{cases}$$

and

$$F^{(i)}_{\lambda-s} = \begin{cases} F^{(i)}_{\lambda-s} & \text{if } \lambda-s \geqslant 0, \\ 0 & \text{if } \lambda-s < 0 \end{cases}$$

($j = 2,3,4,$; $\lambda = 3,4,5,\ldots$; $s = 1,2,3,4$; $i = 0,2$),

the following solution corresponding to $c = 0$, is obtained

$$\overline{W} = \left[1 + \sum_{\lambda=3}^{\infty} a_\lambda R^\lambda\right] a_0 + \left[R^2 + \sum_{\lambda=3}^{\infty} a_\lambda R^\lambda\right] a_2. \qquad (12)$$

It is also evident that the solution corresponding to other values of c is included in equation (12).

4. CONVERGENCE OF THE SOLUTION

The convergence of the solution (12) is tested by applying the technique used by Lamb [4]. Thus, equating to zero the coefficient of the general term occurring in (8) and taking the limit as $\lambda \to \infty$, one has

$$(1-\eta)\mu^4 + \{\eta-2(1-\eta)\beta\}\mu^3 - \{2\eta\beta - (1-\eta)\beta^2\}\mu^2 + \eta\beta^2\mu = 0, \qquad (13)$$

where

$$\mu = \lim_{\lambda\to\infty} \frac{a_{\lambda+1}}{a_\lambda}.$$

The roots of the equation (13) are $\mu = 0, \beta, \beta$ and $-\frac{\eta}{1-\eta}$. Thus in the interval $0 \leqslant R < 1$, when $|\mu| < 1$ and hence for all $|\beta| < 1$ and $\eta < 0.5$, the solution (12) is uniformly convergent.

5. BOUNDARY CONDITIONS AND FRFQUENCY EQUATIONS

The frequency equations for clamped (C) and simply supported (SS) circular plates are obtained by using the following boundary conditions:

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C-PLATES. For a circular plate clamped at the edge $r = a$, the deflection and slope of toe plate element at the edge snould be zero i. e.

$$w(r,t) \mid_{r=a} = \frac{\partial w(r,t)}{\partial t} \mid_{r=a} = 0$$

$$\text{or } \overline{W} \mid_{R=1} = \frac{\partial \overline{W}}{\partial R} \mid_{R=1} = 0. \tag{14}$$

When the boundary conditions (14) are enforced, equation (11) yields the following frequency equation:

$$\begin{vmatrix} Q_1(\Omega) & Q_2(\Omega) \\ Q_3(\Omega) & Q_4(\Omega) \end{vmatrix} = 0, \tag{15}$$

where

$$Q_1(\Omega) = 1 + \sum_{\lambda=3}^{\infty} F_\lambda^{(0)}, \quad Q_2(\Omega) = 1 + \sum_{\lambda=3}^{\infty} F_\lambda^{(2)}, \quad Q_3(\Omega) = \sum_{\lambda=3}^{\infty} \lambda F_\lambda^{(0)}$$

and

$$Q_4(\Omega) = 2 + \sum_{\lambda=3}^{\infty} \lambda F_\lambda^{(2)}.$$

SS-PLATES. For a circular plate simply-supported at the edge $r = a$, the boundary conditions are

$$w(r,t) \mid_{r=a} = M_r(r,t) \mid_{r=a} = 0,$$

or

$$\overline{W} \mid_{R=1} = \left[\frac{\partial^2 \overline{W}}{\partial R^2} + \frac{\nu}{R} \frac{\partial \overline{W}}{\partial R} \right] \mid_{R=1} = 0. \tag{16}$$

Applying the boundary conditions (16) on equation (12) the characteristics equation for the determination of frequencies, is obtained as

$$\begin{vmatrix} Q_1(\Omega) & Q_2(\Omega) \\ Q_5(\Omega) & Q_6(\Omega) \end{vmatrix} = 0, \tag{17}$$

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where

$$Q_5(\Omega) = \sum_{\lambda=3}^{\infty} \lambda(\lambda+\nu-1)\, F_{\lambda}^{(0)} \quad \text{and}$$

$$Q_6(\Omega) = 2(1+\nu) + \sum_{\lambda=3}^{\infty} \lambda(\lambda+\nu-1)\, F_{\lambda}^{(2)} .$$

6. DEFLECTION AND MOMENTS

Usiing the boundary conditions at $R = 1$ with $a_0 = 1$ on the deflection function $\overline{W}$ (12), one finds that the value of a_2 i. e.

$$a_2 = - \frac{1 + \sum_{\lambda=3}^{\infty} F_{\lambda}^{(0)}}{1 + \sum_{\lambda=3}^{\infty} F_{\lambda}^{(2)}} ,$$

Thus after substituting for a_0 and a_2, equation (12) becomes

$$\overline{W} = \left[1 + \sum_{\lambda=3}^{\infty} F_{\lambda}^{(0)} R^{\lambda} \right] - \frac{1 + \sum_{\lambda=3}^{\infty} F_{\lambda}^{(0)}}{1 + \sum_{\lambda=3}^{\infty} F_{\lambda}^{2)}} \left[R^2 + \sum_{\lambda=3}^{\infty} F_{\lambda}^{(2)} R^{\lambda} \right] \quad (18)$$

Applying the boundary conditions at $R = 1$ with same values of a_0 and a_2 on the non-dimensional moment parameter i. e. on $\overline{M} = \frac{M_R}{D_0}$ one gets

$$\overline{M} = - \{1-\eta(1-R)\}\, (1-\beta R)^3 \left[\sum_{\lambda=3}^{\infty} \lambda(\lambda-1+\nu)\, F_{\lambda}^{(0)}\, R^{\lambda-2} \right.$$

$$\left. - \frac{1 + \sum_{\lambda=3}^{\infty} F_{\lambda}^{(0)}}{1 + \sum_{\lambda=3}^{\infty} F_{\lambda}^{(2)}} \{2(1+\nu) + \sum_{\lambda=3}^{\infty} \lambda(\lambda-1+\nu)\, F_{\lambda}^{(2)}\, R^{\lambda-2}\} \right] \quad (19)$$

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where

$$D_0 = \frac{E_0 H_0^3 a_0}{12(1-\nu^2)} .$$

6. DISCUSSION OF RESULTS

In both the cases of edge conditions i.e. clamped and simplysupported of circular plates, the frequeucy parameter $\Omega(= a\omega\sqrt{(1-\nu^2)} / (\overline{E}_0/\overline{\rho}))$, for the first two modes of vibration have been computed for different values of taper constant β and thermal gradient η with fixed values of Poisson's ratio and initial thickness taken as 0.3 and 0.1 respectively. In computing the the frequency parameter, terims upto an accuracy of 10^{-8} on their absolute values in the seres have been retained.

In Figure-1, the effect of linear thickness on the frequency parameter Ω of heated and unheated circular plates, is shown. The values of Ω for $\eta = 0.0$ have been found to decrease 39.81% and 31.37% for a clamped circular plate, 27.61% and 28.56 for SS-plate for the first and second modes of vibration, respectively. On the other hand, other hand, for heated plate, these decrements are 40.31, 31.32 and 28 74. 29 12 percent respectively. Thus, one can conclude from the above percentage the shapes of the curves, for heated and unheated plates for both the edge conditions and modes of vibration considered here, are almost linear.

The effect of thermal gradient η on the frequency parameter Ω, for clamped and simply supported edge conditions for the first two modes of vibration of the uniform and non-uniform circular plates, has been shown in Fig 2 It is found for $\beta = 0.0$, the Ω decreases 7.72% and 9 95% for clamped plate and 10.09% and 10.43 for SS-plate for the first two modes of vibration respectively. When the value

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of β is taken 0.3, the decrement with linearly varying temperature gradient η, are noted 8.02, 10.07 and 10.66, 9.85 percent for the same boundary conditions and modes of vibration considered above. The effect of linear temperature, on the frequencies of uniform and non-uniform clamped circular plates is higher than simply-supported plate, is also noted.

The transverse displacement $\overline{W}$ and moment parameter $\overline{M}$, for the first two modes for both edge conditions, are shown in Figures 3 and 4 resyectively.

To make a comparision with the results in references [3] and [14], the value of Ω was replaced by $\frac{a\omega}{H_0}\sqrt{12(1-\nu^2)}\ \bar{\rho}\ /\ \bar{E}_0$ and the frequencies were computed for the first two modes of vibration for both clamped and simply supported edge conditions, for three values of the paper constant. These results are shown in Table 1. It can be seen from this table that the results are in good agreement.

TABLE 1

A comparison of the natural frequencies of circular plate of linearly varying thickness, $H_0 = 0.1$, $\nu = 0.3$, $\eta = 0.0$

β	Clamped plate		Simply-supported plate	
	First Mode	Second Mode	First Mode	Second Mode
0.1	9.40578	37.37706	4.66706	28.07945
	(9.40578)*	(37.37705)	(4.66707)	(28.07944)
	(9.4058)**	(37.3771)	(4.6671)	(28.0795)
0.3	7.80704	32.47526	4.14675	24.74402
	(7.80705)	(32.47527)	(4.14676)	(24.74401)
	(7.8071)	(32.4753)	(4.1468)	(24.7440)
0.5	6.23385	27.34074	3.63707	21.28774
	(6.23384)	(27.34073)	(3.63707)	(21.28773)
	(6.2339)	(27.3408)	(3.6371)	(21.2878)

* Reference [3]

**Reference [14]

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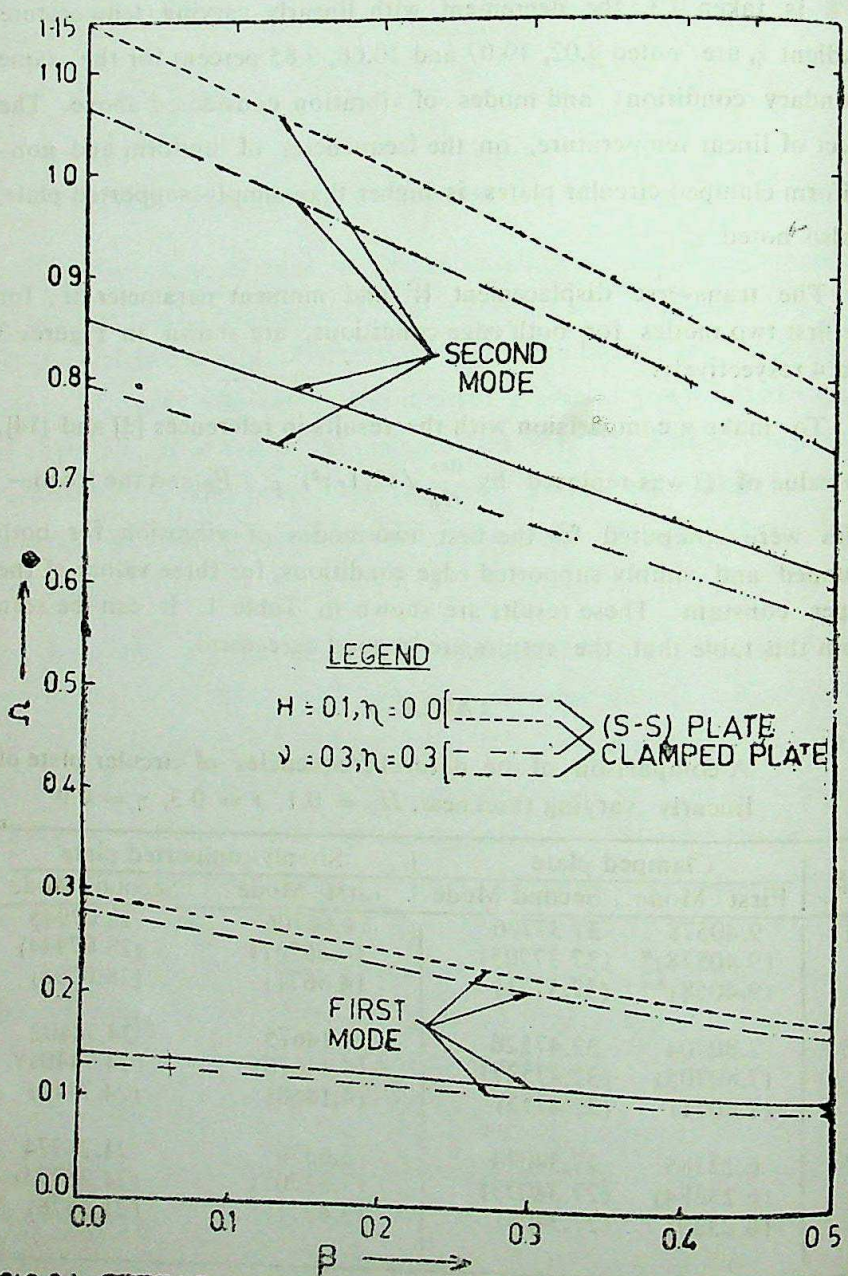

FIG 1 EFFECT OF TAPER CONSTANT β ON THE FREQUENCY PARAMETER 'Ω'

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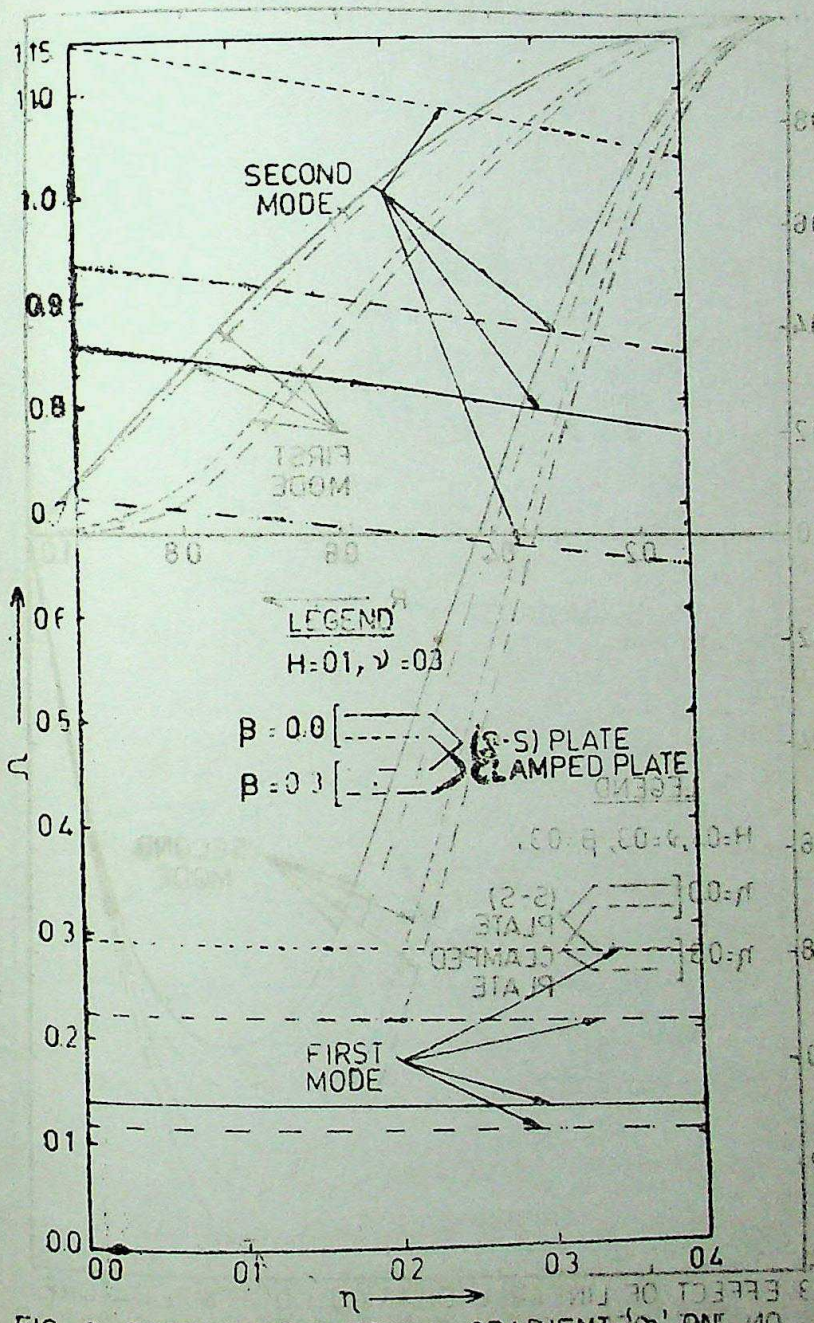

FIG 2 EFFECT OF THERMAL GRADIENT ON THE FREQUENCY PARAMETER

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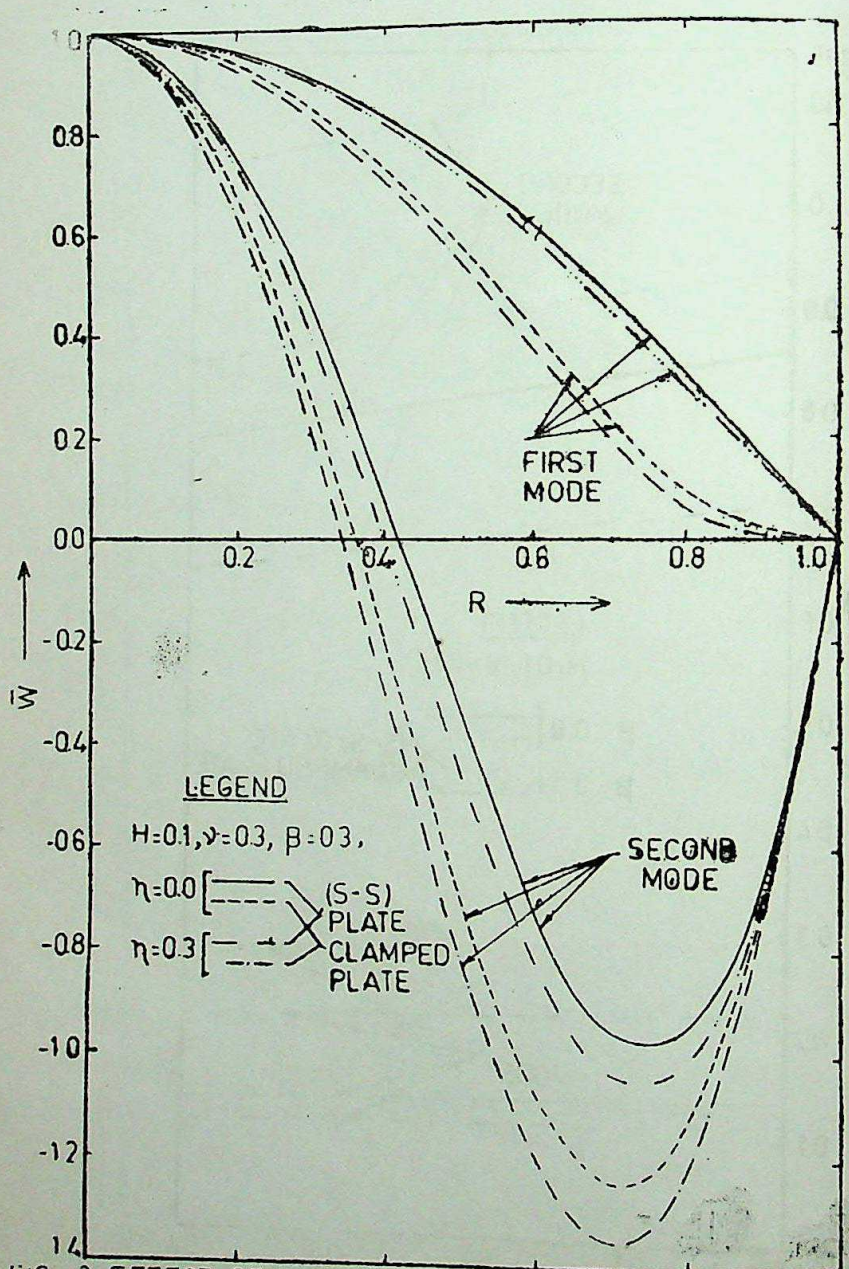

FIG 3 EFFECT OF LINEARLY DISTRIBUTED TEMPERATURE ON THE DEFLECTION PARAMETER '$\bar{W}$' OF CIRCULAR PLATE OF LINEARLY VARYING THICKNESS.

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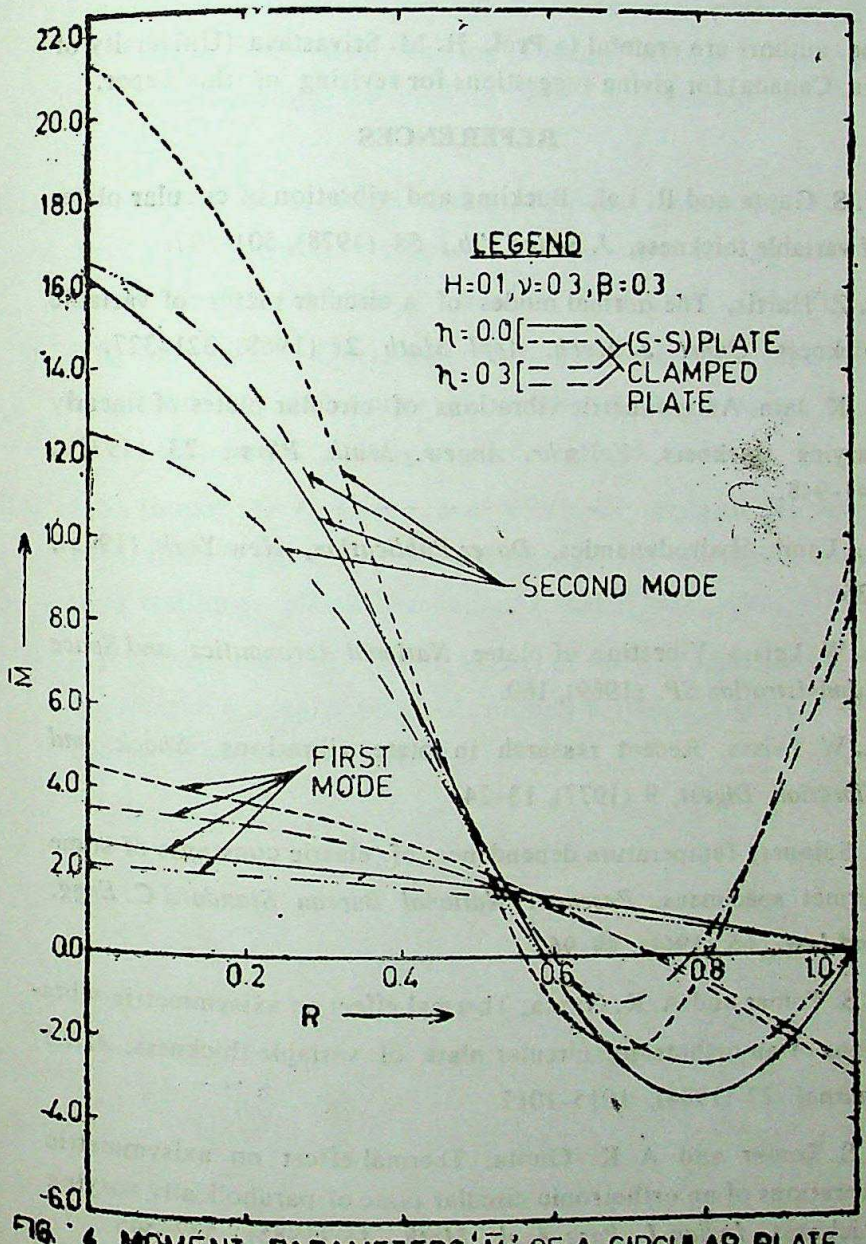

FIG. 4 MOMENT PARAMETERS '$\bar{M}$' OF A CIRCULAR PLATE OF LINEARLY VARYING THICKNESS UNDER THE INFLUENCE OF LINEARLY VARYING TEMPERATURE FIELD.

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Acknowledgements

The authors are grateful to Prof. H. M. Srivastava (University of Victoria, Canada) for giving suggestions for revising of this paper.

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

A FIXED POINT THEOREM IN UNIFORMLY CONVEX SPACES

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(Received: March 30, 1992)

Bogin [1] and Rhoades [8] have generalized a fixed point theorem in uniformly convex spaces of Goebel, Kirk and Shimi [6]. In this paper, to present another extension of their result, we establaih the following theorem;

THEOREM. Let X be a uniformly convex Banach space, K a non-empty bounded closed and convex subset of X, and $F:K\to K$ a continuous mapping satisfying for $x,y \in K$:

$$\begin{aligned}(1)\quad \|F(x)-(y)\| \leqslant\ & \alpha\|x-y\| + \beta\,(\|x-F(x) + \|y-F(y)\|) \\ & + \gamma(\|x-F(y)\| + \|y-F(x)\|) \\ & + \sigma(\|F\,y) - F^2(x)\| + \|x-F^2(x)\| \\ & + \eta(\|F(x) - F^2(x)\| + \|y-F^2(x)\|)\end{aligned}$$

where $\alpha, \beta, \gamma, \sigma$ and η are non-negative real numbers such that;

(2) $\alpha+2(\beta+\gamma+\sigma+\eta) = 1$, $\alpha+\beta+2\gamma+\sigma+\eta < 1$ and $\alpha+2\beta+2\eta < 1$.

Then F has a fixed point in K.

It may be mentioned that the result in [6] is obtained by taking $\gamma = \sigma = 0$ in (1). We shall follow the same line of argument and notation as that of [6].

Proof. Putting $y = F(x)$ in (1), we get

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$$\| F(x) - F^2(x) \| \leqslant \alpha \| x - F(x) \| + \beta (\| x - F(x) + \| F(x) - F^2(x) \|)$$
$$+ \gamma (\| x - F^2(x) \| + \| F(x) - F(x) \|)$$
$$+ \sigma (\| F^2(x) - F^2(x) \| + \| x - F^2(x) \|)$$
$$+ \eta (\| F(x) - F^2(x) \| + F(x) - F^2(x) \|)$$
$$\leqslant (\alpha+\beta) \| x - F(x) \| + (\beta+2\eta) \| F(x - F^2(x) \|$$
$$+ (\gamma+\sigma) \| x - F^2(x) \|$$
$$\leqslant (\alpha+\beta) \| x - F(x) \| + (\beta+2\eta) \| F(x) - F^2(x) \|$$
$$+ (\gamma+\sigma) (\| x - F(x) \| + F(x) - F^2(x) \|)$$

$$\| F(x) - F^2(x) \| \leqslant \frac{\alpha+\beta+\gamma+\sigma}{1-\beta-\gamma-\sigma-2\eta} \| x - F(x) \|.$$

By (2), $\frac{\alpha+\beta+\gamma+\sigma}{1-\beta-\gamma-\sigma-2\eta} = 1$. Therefore

(3) $\| F(x) - F^2(x) \| \leqslant | x - F(x) |$,

and consequently

(4) $\| F^{i+1}(x) - F^i(x) \| \leqslant F^i(x) - F^{i-1}(x) \|$ $i = 1, 2, 3 \ldots$

Assertion: $\inf_{x \in K} \| x - F(x) \| = 0$.

To prove the assertion, we assume $\inf_{x \in K} \| x - F(x) \| = d > 0$.

Let $\epsilon > 0$ and choose $x \in K$ such that $\| x - F(x) \| \leqslant d + \epsilon$ using uniform convexity and asumptions on K, we can find a real number p, $0 < p < 1$, as in [6] such that

(5) $\| F^{i-1}(x) - F^{i+1}(x) \| \leqslant 2p \| F^{i-1}(x) - F^i(x) \|$, $0 < p < 1$

Writing j for $i-1$ in (5), we obtain

(6) $\| F^j(x) - F^{j+2}(x) \| \leqslant 2p \| F^j(x) - F^{j+1}(x) \|$, $0 < p < 1$

Case 1. γ or $\sigma \neq 0$.

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By (1)

$$\|F^i(x)-F^{i+1}(x)\| \leqslant(\alpha+\beta)\|F^i(x)-F^{i-1}(x)\| + (\beta+2\eta)\|F^i(x)-F^{i-1}(x)\|$$
$$+ (\gamma+\sigma)\|F^{i+1}(x)-F^{i-1}(x)\|$$

Therefore by (5),

$$(1-\beta-2\eta)\|F^i(x)-F^{i+1}(x)\| \leqslant (\alpha+\beta+2p(\gamma+\sigma))\|F^i(x)-F^{i-1}(x)\|$$

hence $(1-\beta-2\eta, d \leqslant (\alpha+\beta+2p(\gamma+\sigma))(d+\epsilon)$.

Since $\epsilon > 0$ is arbitrary, we have

$$(1-\beta-2\eta)\, d < (\alpha+\beta+2p(\gamma+\sigma))\, d.$$

This implies

$$(1-\beta-2\eta \leqslant \alpha+\beta+2p(\gamma+\sigma)$$

which, in view of $p<1$ contradicts (2).

Case II. $\gamma=\sigma=0$ Let

$m=[F^i(x)+F^{i+1}(x)]/2$. Then by (1),

$$\|m\ F(m)\| \leqslant 2^{-1}\|F^i(x)-F(m)\| + 2^{-1}\|F^{i+1}(x)-F(m\ \|$$
$$\leqslant 2^{-1}[\alpha\|F^{i-1}(x)-m\|+\beta(\|F^{i-1}(x)-F^i(x)\|+\|m-F(m)\|]$$
$$+\eta[\|F^i(x)\ F^{i+1}(x)\|+\|m-F^{i+1}(x)\|)]$$
$$+2^{-1}[\alpha\|F^i(x)-m\|+\beta\|F^i(x)-F^{i+1}(x)\|+\|m-F(m)\|]$$
$$+\eta[\|F^{i+1}(x)-F^{i+2}(x)\|+\|m-F^{i+2}(x)\|)].$$

Therefore

$$(1-\beta)\|m\ F(m)\| \leqslant 2^{-1}[\alpha(\|F^{i-1}(x)-m\|+\|F^i(x)-m\|)$$
$$+\beta(\|F^{i-1}(x)-F^i(x\ \|+\|F^i(x)-F^{i+1}(x)\|)$$
$$+\eta\ \|F^i(x)-F^{i+1}(m)\|+\|m-F^{i+1}(x)\|+\|F^{i+1}(x)-F^{i+2}(x)\|$$
$$+\|m-F^{i+2}(x)\|)]$$

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$$\leqslant 4^{-1}(\|F^{i-1}(x)-F^{i}(x)\|+\|F^{i-1}(x)-F^{i+1}(x)\|$$

$$+\|F^{i}(x)-F^{i+1}(x)\|)+2^{-1}[\beta\ \|F^{i-1}(x)-F^{i}(x)\|$$

$$+\|F^{i}(x)-F^{i+1}(x)\|)+\eta\ \|F^{i}(x)-F^{i+1}(x)\|$$

$$+\|F^{i+1}(x)-F^{i+2}(x)\|+2^{-1}(\|F^{i}(x)-F^{i+1}(x)\|$$

$$+\|F^{i+1}(x)-F^{i+2}(x)\|+\|F^{i}(x)-F^{i+2}(x)\|)],$$

Hence applying (5) and (6) we have

$$(1-\beta)d<\tfrac{1}{4}(\alpha+2\alpha p+\alpha+\eta+\eta+2\eta p)\ (d+\varepsilon)+\tfrac{1}{2}(\beta+\beta+\eta+\eta)\ (d+\varepsilon)$$

letting $\epsilon\to 0$ we obtain

$$(1-\beta)d\leqslant\tfrac{1}{4}(2\alpha\ (1+p)\ +\ 2\eta(1+p))\ d\ +\ (\beta+n)d$$

that is, if α or $\eta\neq 0$,

$$(1-\beta)<\alpha+\beta+2\eta,$$

which contradicts (2).

Case III. $\alpha=\gamma=\sigma=\eta=0$

In view of Soardi's result [9] this case need not be considered. Thus the assertion is proved.

Now for $\varepsilon\in(0,1)$ assume

$$C_\varepsilon=\{x:\ |\ x-F\ x)\ ||\leqslant\varepsilon\}\ \text{and}$$

$$D_\varepsilon=\{x\in C_\varepsilon:\ \|\ x\ \|\leqslant\bar{a}+\varepsilon\},$$

where $\bar{a}=\lim_{\varepsilon\to 0} a, C_\varepsilon)$ and $a\ C_\varepsilon)=inf\ \{\|x\|:x\in C_\varepsilon\}$

Since F is continuous the sets C_ε and $(S_0)\ D_\varepsilon$ are closed and by the assertion they are non-void. The proof is completed by showing

$$\bigcap_{\varepsilon>0} C_\varepsilon\neq\phi.$$

If $\bar{a}=0$ then $0\in\bigcap_{\varepsilon>0} C_\varepsilon\neq\phi$. Therefore we may assume $\bar{a}>0$,

letting $u_1, u_2\in C_\varepsilon$ we have by (1) and the triangle inequality for $i=1,2$.

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$$(7)\quad \|u_1-F((u_1+u_2)/2\| \leqslant \|u_1-F(u_1)\| + \|F(u_1)-F(u_1+u_2)/2)\|$$

$$\leqslant \epsilon + \alpha\|u_1-(u_1+u_2)/2\| + \beta(\|u_1-F(u_1)\|$$

$$+ \|(u_1+u_2)/2-F((u_1+u_2)/2\|) + \gamma(\|u_1-F((u_1+u_2)/2\|$$

$$+ \|(u_1+u_2)/2-u_1\| + \|u_1-F(u_1)\|) + \sigma(\|u_1-F(u_1)\|$$

$$+ \|F((u_1+u_2)/2)-u_1\| + \|u_1-F(u_1)\| + \|F(u_1)-F^2(u_1)\|)$$

$$+ \eta(\|F(u_1)-F^2(u_1)\| + \|(u_1+u_2)/2-u_1\|$$

$$+ \|u_1-F(u_1)\| + \|F(u_1)-F_2(u_1)\|)$$

We note by (3) that

$$\|F(u_1)-F^2(u_1)\| \leqslant \|u_1-F(u_1)\| \leqslant \varepsilon.$$

Also since

$$\|(u_1+u_2)/2-F((u_1+u_2)/2)\| \leqslant \max_{i=1,2} \|u_i-F((u_1+u_2)/2)\|.$$

We obtain

$$(1-\beta-\gamma-\sigma) \max_{i=1,2} \|u_i-F((u_1+u_2)/2)\|$$

$$\leqslant (1+\beta+\gamma+3\sigma+3\eta)\,\varepsilon + (\alpha+\gamma+\eta)\,\|u_1-u_2\|/2$$

So $\|u_1-F((u_1+u_2)/2\| \leqslant q\varepsilon + r\|u_1-u_2\|/2,$

where $q = (1+\beta+\gamma+3\sigma+3\eta) / (1-\beta-\gamma-\sigma)$ and $r = (\alpha+\gamma+\eta)/(1-\beta-\gamma-\sigma)$.

By (2), $r < 1$, therefore

$$\|u_1-F((u_1+u_2)/2\| \leqslant q\varepsilon + \|u_1-u_2\|/2.$$

Now the rest part of proof follows as in [6] we have remarked that if β or σ or $\eta \neq 0$ in (2) then the fixed point is unique.

REMARK: If X is compact metric space and F a continuous mapping of X into itself, satisfying the inequality,

$$d(Fx,Fy) < \alpha d(x,y) + \beta\{d(x,Fx) + d(y,Fy)\} + \gamma\{d(x,Fx) + d(y,Fx)\}$$

$$+ \sigma\{d(Fy,F^2x) + d(x,F^2x)\} + \eta\{d(Fx,F^2x) + d(y,F^2x)\},$$

where α, β, γ, σ and η are non-negative real numbers such that, $\alpha + 2(\beta+\gamma+\sigma+\eta) \leqslant 1$ and $\gamma+\sigma+\eta > 0$ Then F has a unique fixed point.

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We further obtain from this remark:

(i) A result of Khan and Sharma [7] if we put $\eta=0$.

(ii) A result of Edelstein [2] if we put $\beta = \gamma = \sigma = \eta = 0$ and $\alpha = 1$.

(iii) A result of Fisher [3] if we put $\alpha = \gamma = \sigma = \eta = 0$ and $\beta = \frac{1}{2}$.

(iv) A result of Fisher [4] if we put $\alpha = \beta = \sigma = \eta = 0$ and $\gamma = \frac{1}{2}$.

(v) A result of Fisher [5] if we put $\alpha = a$, $\beta = b$, $\gamma = c$ and $\eta = \sigma = 0$.

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

APPLICATIONS OF FIXED POINT THEOREMS

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Several interesting results using fixed point theory are given in Approximation Theory. An excellent reference is Cheney [5]. For a survey paper one is referred to [9]. The following theorem, due to Brosowski [3], has been a basic important result.

Let f be a contractive linear operator of a normed linear space X. Let C be an f-invariant subset of X and x an f-invariant point. If the set of best C-approximants to x is nonempty, compact and convex, then it contains an f-invariant point. Subrahmanyam [13], S. P. Singh [10], [11] anp K. L. Singh [9] deal with fixed point theorems of nonexpansive mapppings.

Carbone [4] and Rao and Mariadoss [7] deal with generalized nonexpansive mappings Sahab, Khan and Sessa [8] extended the result of [10] in a different direction whereas Hicks and Humphries relaxed the boundary condition in their paper [6]. The purpose of this note is to give some further results. We give the following definitions.

Let X be a normed linear space and C be a nonempty subset of X. For any $x \in X$, we define

$$d(x,C) = inf \{ \| x-y \| : y \in C\}$$

and an element $y \in C$ is called a best approximation of x by means of elements of C if

$$d(x,C) = \|x-y\|.$$

The set

$$\{y \in C: \|x-y\| = d\ x,C)\}$$

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is denoted by $P_C(x)$. The set valued mapping $P_C(x)$ is called the *metric projection* associated with C. If $P_C(x)$ is nonempty for every $x \in X$, then C is called *proximinal* The set C is called *Chebyshev set* if the mapping $x \to P_C(x)$ is single-valued; that is, $P_C(x)$ contains at most one element for every $x \in X$.

A function f defined on a convex set C is called *convex* if

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y)$$

$0 \leqslant \lambda \leqslant 1$. If f is convex then the set

$$A + \{x \in C / f(x) \leqslant \alpha, \alpha \in R\}$$

is convex.

Let $x_1, x_2 \in A$. We need to show that

$$\lambda x_1 + (1-\lambda)x_2 \in A.$$

Since f is convex, we get

$$f(\lambda x_1 + (1-\lambda)x_2) \leqslant \lambda f(x_1) + (1-\lambda) f x_2) \leqslant \lambda\alpha + (1-\lambda)\alpha = \alpha.$$

Thus

$$\lambda x_1 + (1-\lambda)x_2 \in A$$

i.e., A is convex.

However, the converse is not true.

A convex function need not be continuous.

The first result using convex functions for fixed point theory was given by Belluce and Kiak in 1968 [2].

THEOREM B. *Let C be a nonempty, weakly compact subset of a Banach space X and let T: $C \to C$ be continuous. If*
1. *$(I-T)$ is convex and* 2. *in $f \| (I-T)x \| = 0$,*
then T has a fixed point.

A functiou f on a convex set C is called *quasi-convex* if

$$f(\lambda x + (1-\lambda)y) \leqslant max\,(f(x), f(y)).$$

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Quasi-convex functions are more general then convex functions. For example, let us consider $C = (0,\infty)$ and $f(x) = \sqrt{x}$.

Then clearly f is quasi-convex, but f is not convex. Taking $x = 1, y = 4$, we get

$$f\left(\frac{x+y}{2}\right) = f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^{\frac{1}{2}},$$

while

$$\frac{1}{2}(f(x) + f(y)) = \frac{1}{2}(1+2) = \frac{3}{2} < \left(\frac{5}{2}\right)^{\frac{1}{2}}.$$

A function f is *quasi-convex* on C if and only if

$$A = \{x \in C / f(x) \leqslant \alpha\} \quad \text{is convex.}$$

Proof Let $x_1, x_2 \in C$ and $f(x_2) \leqslant f(x_1)$, $0 \leqslant \lambda \leqslant 1$.
Then by the convexity of A we get

$$f((\lambda x_1 + (1-\lambda)x_1) \leqslant \alpha = f(x_1).$$

This implies that f is quasi-convex on C.

In case f is quasi-convex, α is any real number, let, $x_1, x_2 \in A$ and let $f(x_2) \leqslant f(x_1)$. Since $x_1, x_2 \in A$, $f(x_2) \leqslant f(x_1) \leqslant \alpha$. f quasi-convex and C is convex we have, for $0 \leqslant \lambda \leqslant 1$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leqslant f(x_1) \leqslant \alpha$$

Hence, $(1-\lambda)x + \lambda x_1 \in A$, i.e., A is convex.

In case $f(\lambda x_1 + (1-\lambda)x_2) \leqslant max\,(f(x_1), f(x_2))$,
the f is also called *strongly quasi-convex function.*

The condition

$$f\left(\frac{x+y}{2}\right) \leqslant max\,(f(x), f(y))$$

always implies uniqueness; i.e., a strongly quasi-convex function can achieve its minimum over a convex set C at no more than one point.

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In fact, if

$$f(x_0) = f(x_0) = \inf_{x \in C} f(x),$$

then

$$f\left(\frac{x_0 + y_0}{2}\right) \leqslant max\,(f(x_0), f(y_0));$$

contradiction

The following result is due to Singh and Veitch [12].

Theorem S - *Let C be a nonempty, weakly compact, convex subset of a Banach space X If T: $C \to C$ is continuous, $(I-T)$ is quasi convex on C and*

$$in\ f_{x \in C} \|(I-T)x\| = 0,$$

then T has at least one fixed paint.

We incldue the proof for the sake of completeness.

Proof If a quasi-convex functional defined on a weakly closed convex set C is lower semicontinuous, then it is weakly lower semicontinuous.

Consider

$$J(x) = \|(I-T)x\|.$$

$I-T$ quasi-convex implies that J is quasi convex $I-T$ lower semicontinuous hence J is weakly lower semicontinuous. Since C is weakly compact J attains its minimum. i.e., there is an $x_0 \in C$ such that

$$\|(I-T)x_0 = \inf_{x \in C} \|(I-T)x\|.$$

But $\inf_{x \in C} \|(I-T)x\| = 0$, therefore $\|(I-T)x_0\| = 0$.

So $\qquad Tx_0 = x_0.$

Corollary 1. *Let C be a nonempty weakly compact convex subset of a Banch space X. If : $C \to C$ is nonexpansive map and $(I-T)$ is quasi-convex, then f has a fixed point.*

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Remark. Since a nonexpansive map is continuous and

$$inf\ \{\|(I - f)x\|\ x \in C\} = 0,$$

so the result follows from Theorem S.

Theorem 1. *Let C be a subset of a Banach space X and $T: X \to X$ a map such x_0 is invariant under T and $T(\partial C) \subseteq C$. If the set D of best C-approximants to x_0 is weakly compact, convex, T is nonexpansive on $D \cup \{x_0\}$ and $(I-T)$ is quasiconvex, then T has a fixed point closest to x_0.*

Proof. Fist of all we prove that $T: D \to D$

In fact, if $y \in D$, then y is closest to x_0.

Now

$$\|Ty - x_0\| = \|Ty - Tx_0\| \leqslant \|y - x_0\|.$$

This implies that $Ty \in D$.

Since D is nonempty, weakly compact, convex, $T: D \to D$ is nonexpansive and $(I\ T)$ is quasiconvex so T has a fixed point in D closest to x_0.

Note. In Theorem 1 we do not require that T is nonexpansive and linear as in [4], and $T: C \to C$ is not needed.

Theorem 2. *Let X be a Banach space and $T: X \to X$ a map such that $Tx_0 = x_0$ and $T(\partial C) \subseteq C$.*

Further, if the set D of best C-approximants to x_0 is closed bounded and convex, T is a nonexpansive map on $D \cup \{x_0\}$ and $(I\ T)(D)$ is closed then T has a fixed point closest to x_0

For our proof, we need the following

Theorem : *Let X be Banach space and C a closed bounded convex subset of X. If $f: C \to C$ is nonexponsive and $(I - f)\ C$ is a closed of X, then f has a fixed point in C.*

Proof It is easy to show as in Theorem 1 that $T: D \to D$.

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T has a fixed point in D by the above theorem, say $Tu = u$, $u \in D$. Clearly, u is closest to x_0.

Note. In the above Theorem 2, T need not be even continuous on X. T should have desired conditions to garantee a fixed point only on D.

We need the following in our Theorem 3.

Theorem B ([1]) *Let $C \neq \Phi$ weakly compact convex subset of a Banach space X and $f: C \to C$ a nonexpansive mapping. If there is a strictly increasing function*

$$g: [0,\infty) \to [0,\infty)$$

with $g(0) = 0$, such that for all $x, y \in C$ and $0 \leqslant \lambda \leqslant 1$, we have

$$\| f(\lambda x + (1-\lambda)y) - \lambda x - (1-\lambda)y \| \leqslant g(\|x - fx\| + \|y - fy\|),$$

which is weaker than convexity condition.

Then f has a fixed point.

Theorem 3. *Let X be a Bœnach space and $T: X \to X$ a map. Let $Tx_0 = x_0$ and $T(\partial C) \subset C$. Assume that the set D of best C-approximants to x_0 is nonempty, weakly compact, convex and T is nonexpansive on $D \cup \{x_0\}$. Furthermore, if there is a strictly increasing function $g:[0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that for all $x, y \in C$ and $0 < \lambda < 1$, we have*

$$\|T(\lambda x + (1-\lambda)y) - \lambda x - (1-\lambda)y\| \leqslant g(\|x - Tx\| + \|y - Ty\|),$$

then T has a fixed point closest to x_0.

Proof. It is easy to show that $T: D \to D$, and then using the above T heorem B we get that T has a fixed point in D, closest to x_0.

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

EXISTENCE AND UNIQUENESS FOR A NONLINEAR TWO POINT BOUNDARY VALUE PROBLFM

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(Received: July 2, 1992)

ABSTRACT

In this paper we study a nonlinear two point boundary value problem. We prove an uniqueness theorem, an existence theorem of constructive type, and an existence and uniqueness theorem of the solution of the two point boundary value problem, using iteration processes and fixed point methods in a Hilbert space.

1. INTRODUCTION

This paper concerns the following nonlinear two point boundary value problem:

Find $u(t) \in H^2[0.1]$ such that

$$(1.1) \quad \begin{cases} u''(t) + \lambda f(u(t)) = 0 \\ u(0) = u(1) = 0, \end{cases}$$

where $H^2[0,1]$ is the completion of $C^2[0,1]$ with respect to the L^2 norm. Our object is to obtain an uniqueness theorem, an existence theorem of constructive type and, as corollary, an existence and uniqueness theorem of the solution of problem (1.1) for λ fixed in some range. For simplicity, we indicate the main result of this paper, assuming f to be a twice differentiable function. If f satisfies these conditions

(i) $f(t) > 0$

(ii) $0 \leqslant f'(t) \leqslant L$

(iii) $f''(t) < 0$

for some $L > 0$ and for all $t \geqslant 0$ then problem (1.1) has an unique solution for

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$$0 < \lambda < \frac{\pi^2}{L}, \lambda \neq \frac{1}{LK}$$

where $K^2 = \int_0^1 \int_0^1 G^2(x,t)\, dtdx = \frac{1}{90}$ and $G(x,t)$ is the triangular kernel. Moreover the solution is given by an iterative sequence. The existence result is obtained from the following fixed point theorem. Let H a Hilbert space and $T: H \to H$ an operator. If T is monotone and Lipschitz continuous operator with constant $\beta > 1$ then the following sequence

$$\begin{cases} x_0 \in X \\ x_{n+1} = x_n + \mu(Tx_n - x_n), \end{cases}$$

where $0 < \mu < \frac{2}{\beta - 1}$, converges to the unique fixed point of T.

The paper is organized as follows. Section 2 is devoted to the introduction of some basic results on a two point boundary value problem. Section 3 contains the proof of a fixed point theorem in Hilbert spaces. Finally in section 4 we prove an uniqueness theorem of the solution, an existence theorem of the solution of constructive type, and as corollary, an existence and uniqueness theorem, for a class of nonlinear two point boundary value problems.

2. **Preliminary facts**. In this section we assume that $f: R \to R^+$ is a positive and locally Lipschitz continuous function.

Proposition 2.1 *All solutions of (1 1) are strictly positive on* $(0,1)$ *for* $\lambda > 0$.

Proof. Since f and λ are both positive, then we must have $u''(x) < 0$ for all $x \in (0,1)$. Taylor's expansion of solution $u(x)$ near $x = 0$ gives us

$$u(x) = xu'(0) + \frac{1}{2}x^2 u''(\xi_x),\ 0 < \xi_x < 1.$$

Letting $x = 1$, we deduce that

$$u(1) = u'(0) + \frac{1}{2} u''(\xi_1)$$

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and, hence, that $u'(0) > 0$. Now we have

$$u(x) = ax + O(x), \quad x \to 0 \quad (a > 0)$$

and therefore there exists c, $0 < c < 1$, such that $u(x) > 0$ on $(0,1)$. Let

$$\gamma = sup\{c: u(x) > 0 \text{ on } (0,c)\}.$$

If $\gamma = 1$ then our statement is proved. Assuming, by contradiction, $\gamma < 1$, necessarily it must be that $u(\gamma) = 0$. Rolle's theorem assures that there exist c_1, c_2 such that

$$0 < c_1 < \gamma < c_2 < 1$$

and

$$u'(c_1) = u'(c_2) = 0.$$

The same argument applied to function $u'(x)$ on interval (c_1, c_2), implies that there exists c_3 $c_1 < c_3 < c_2$ such that

$$u''(c_3) = 0.$$

This is a contradiction, since $u''(x) < 0$ on $(0,1)$.

Proposition 2.2 *Any solution of (1.1) has exactly one maximum on* $(0,1)$.

Proof We prove that by contradiction, assuming that there exists a solution $u_*(x)$ having two maxima, say a and b $(a<b)$ such that

$$u_*(a) = u_*(b) = \max_{x \in [0,1]} u_*(x).$$

Rolle's theorem applied to $u'_*(x)$ on (a,b) implies that there exists a $c \in (a,b)$ such that $u''_*(c) = 0$. We have a contradiction and hence the proposition is proved.

Proposition 2.3 *Any solution of* (1.1) *is symmetric with respect to* $x = \frac{1}{2}$ *and, moreover,*

$$\max_{x \in [0,1]} u(x) = u(\tfrac{1}{2}) \geq \|u(x)\|_{L^2[0,1]}.$$

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Proof. We claim that any solution $u(x)$ of (1.1) satisfies

$$u(1-x) = u(x).$$

It is easy to verify that function $y(x) = u(1-x)$ is a solution of (1.1) too. Indeed, we have

$$y' = -u'(1-x), \quad y'' = u'(1-x)$$

$$y(0) = u(0) = 0, \quad y(1) = u(1) = 0$$

and therefore

$$\begin{cases} y''(x) + \lambda f(y(x)) = 0 \\ y(0) = y(1) = 0. \end{cases}$$

Now we have to prove that $y(x) = u(x)$. Remark that it must be

$$y'(\tfrac{1}{2}) = u'(\tfrac{1}{2}) = 0.$$

Let $r = y(\frac{1}{2}) = u(\frac{1}{2})$. Since, f is locally Lipschitz continuous, then the initial value problem

$$\begin{cases} u'' + \lambda f(u) = 0 \\ u(\frac{1}{2}) = r \\ u'(\frac{1}{2}) = 0 \end{cases}$$

has a unique solution, then it follows that $y(x) = u(x)$.
Taking in account Propositions 2.1 and 2 2, we see that

$$\max_{x\in[0,1]} u(x) = u(\tfrac{1}{2}) \geqslant \|u(x)\|_{L^2[0,1]}.$$

Proposition 2.4 *If u_1, u_2 are distinct solutions for a fixed λ, then either $u_1(t) < u_2(t)$ for $0 < t < 1$ or $u_1(t) > u_2(t)$ for $0 < t < 1$.*

Proof Let $u_1(t)$ and $u_2(t)$ be two solutions for a fixed λ. Denoting $r_i = \max_{t\in[0,1]} u_i(t)$, $i = 1,2$ it must be that $r_1 \neq r_2$ in view of Prop. 2.3. Assume $r_1 < r_2$ and prove that $u_1(t) < u_2(t)$ for $0 < t < 1$. Integrating equation

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$$u_i''(t) + \lambda f(u_i(t)) = 0$$

over $(t, \frac{1}{2})$, we obtain

$$\tfrac{1}{2} u_i'^2(t) + \lambda F(u_i(t)) = \lambda F(r_i), \; i = 1,2$$

where $F(\xi) = \int_0^{\xi} f(x)\,dx$. Since F is increasing $F(r_1) < F(r_2)$ and consequently

$$\tfrac{1}{2} u_1'^2(0) = \lambda F(r_1) < \lambda F(r_2) = \tfrac{1}{2} u_2'^2(0)$$

therefore we deduce

$$u_1'(0) < u_2'(0).$$

Thus we have eiher $u_1(t) < u_2(t)$ for $0 < t \leqslant \frac{1}{2}$ or there exists a smallest c, $0 < c < \frac{1}{2}$ such that $u_1(c) = u_2(c)$. Equations

$$\tfrac{1}{2} u_i' = \lambda(F(r_i) - F(u_i)), \; i = 1,2$$

become

$$u_i' = (2\lambda)^{\frac{1}{2}} (F(r) - F(\omega))^{\frac{1}{2}}, \text{ for } 0 \leqslant t \leqslant \tfrac{1}{2},$$

since u is increasing on $0 \leqslant t \leqslant \frac{1}{2}$. Then we have

$$\int_0^{u_i(t)} \frac{d\omega}{(F(r) - F(\omega))^{1/2}} = t(2\lambda)^{1/2}.$$

Equality $u_1(c) = u_2(c)$ implies that

$$(F(r_1) - F(r_2))^{\frac{1}{2}} = (F(r_2) - F(u))^{\frac{1}{2}}$$

hence $F(r_1) = F(r_2)$. Thus it must be $r_1 = r_2$ against hypothesis $r < r_2$

3. A fixed point theorem.

We recall some prelimiary definitons.

Definition 3.1 *An operator $T: Y \subseteq H \to H$ is called monotone if for all $x, y \in Y$*

(3.3) $\quad (Tx - Ty, x - y) \geqslant 0.$

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Definition 3.2 *An operator $T: Y \subseteq H \to H$ is called Lipschitz continuous if there exists a $\beta > 0$ such that for all $x, y \in Y$*

(3.2) $\|Tx - Ty\| \leqslant \beta \|x - y\|$.

Now we state and prove the main result of this section.

Theorem 3.3 *Let H a Hilbert space, endowed with the inner product $(,)$ and let $T: H \to H$ be a monotone and Lipschitz continuous operator with $\beta > 1$. Then the sequence*

$$\begin{cases} x_0 \in H \\ x_{n+1} = x_n + \mu(Tx_n - x_n), \end{cases}$$

where $0 < \mu < \dfrac{2}{\beta - 1}$, converges to a fixed point of T.

Proof. First of all we remark that

$$x_{n+1} = (1 - \mu)x_n + \mu Tx_n.$$

Let us prove, by induction, that $x_n \in H$ for all n. By hypothesis $x_0 \in H$. Suppose that $x_n \in H$ for a fixed n. Since H is convex x_{n+1} belongs to H.

To study the sequence $\|x_{n+1} - x_n\|$ we start from the following identity:

$$\|x_{n+1} - x_n\| = \|(1-\mu)(x_n - x_{n-1} + \mu(Tx_n - Tx_{n-1}\|$$

which, using the inner product, becomes

$$\|x_{n+1} - x_n\|^2 = (1 - \mu)^2 \|x_n - x_{n-1}\|^2 + 2(1 - \mu)\mu(Tx_n - Tx_{n-1}, x_n - x_{n-1}) + \mu^2 \|Tx_n - Tx_{n-1}\|^2.$$

Applying Schwarz's inequalities and taking in account definition (3.2) and (3.3) we have

$$(Tx_n - Tx_{n-1}, x_n - x_{n-1}) \leqslant \|Tx_n - Tx_{n-1}\| \, \|x_n - x_{n-1}\| \leqslant \beta \|x_n - x_{n-1}\|^2.$$

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It follows for all $n \geqslant 1$ that

$$\|x_{n+1} - x_n\|^2 \leqslant q(\mu)\|x_n - x_{n-1}\|^2$$

where

$q(\mu) = (1-\mu)^2 + 2\beta(1-\mu)\mu + \beta^2\mu^2 = (1-\beta)^2\mu^2 - 2(1-\beta)\mu + 1$ is a quadratic form of variable μ having discriminant $\Delta = 0$. From assumption

$$< \mu < \frac{2}{\beta - 1}$$

it follows that

$$0 < q(\mu) < 1.$$

We can prove, by induction, that

$$\|x_{n+1} - x_n\| \leqslant q(\mu)^{(n-1)/2}\|x_1 - x_0\|$$

for all natural number $n \geqslant 1$ and, hence to conclude that $(x_n)_{n \in N}$ is a Cauchy sequence. Since a Hilbert space is a complete metric space, letting

$$x_* = \lim_{n \to \infty} x_n$$

and taking the limit on both sides of

$$x_{n+1} = x_n + \mu(Tx_n - x_n)$$

we obtain

$$x_* = x_* + \mu(Tx_* - x_*).$$

Since $\mu > 0$, therefore we have $Tx_* = x_*$. Thus operator T has at a least a fixed point.

4. Existence and uniqueness for a point boundary value problem.

We are ready to solve problem (1.1) giving a constructive theorem of exitence and uniqueness of the solution under suitable hypotheses. We start with a general uniqueness of the solution.

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Theorem 4.1 *If f is a continuous positive and concave function, then the nonlinear two point boundary value problem*

$$(1.1)\quad \begin{cases} u'' + \lambda f(u) = 0 \\ u(0) = u(1) = 0 \end{cases}$$

has at most one positive solution for all $\lambda \geq 0$.

Proof. Suppose, by contradiction, that there exist two positive solutions u_1, u_2 for some $\lambda > 0$. By Proposition 2.4 these solutions have to be ordered, say $u_1 > u_2$. Remark that function $v = u_1 - u_2$ satisfies the following linear two point boundary value problem

$$(4.11)\quad \begin{cases} v'' + \lambda\psi(t)v = 0 \\ v(0) = v(1) = 0 \end{cases}$$

where

$$\psi = \psi(t) = \frac{f(u_1(t) - f\ u_2(t))}{u_1(t) - u_2(t)}.$$

We have

$$(4.1.2)\quad \begin{cases} -v'' > \lambda\psi(t)v \\ -u_i'' = \lambda f(u_i),\ i = 1.2. \end{cases}$$

Multiplying (4.1.2) by u_i and v respectively and integrating any one by parts we get

$$-\int_0^1 v''\, u_i dt > \lambda\psi \int_0^1 v u_i' dt,$$

$$\int_0^1 v'\, u_i' > \lambda \int_0^1 v u_i dt$$

and

$$\int_0^1 u_i'\, v dt = \lambda \int_0^1 f(u_i) dt \quad i = 1,2.$$

From these equalities

$$\lambda \int_0^1 f(u_i) v dt = \int_0^1 u_i' v'\, dt = \lambda\psi \int_0^1 v u_i dt$$

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since $\lambda \geqslant 0$ and $v > 0$, it follows that

$$\lambda \int_0^1 (f(u_i) - \psi u_i) v dt = 0$$

and, recalling the definition of ψ, therefore

$$\frac{f(u_i)}{u_i} = \frac{f(u_1) - f(u_2)}{u_1 - u_2}, \quad i = 1,2.$$

On the other hand, since f is concave it must be

$$\frac{f(u_1) - f(u_2)}{u_1 - u_2} \leqslant \frac{f(u_1) - f(0)}{u_1} \leqslant \frac{f(u_2) - f(0)}{u_2}.$$

We deduce that

$$\frac{f(u_i) - f(0)}{u_i} \geqslant \frac{f(u_i)}{u_i}$$

and therefore $f \leqslant (0)$, against hypothesis $f(0) > 0$. We have reached a contradiction: uniqueness of the solution to problem (1.1) is proved.

Theorem 4.2 *Assume that f is continuous function satisfying:*

there exists $L > 0$ such that

(i) $f(t) > 0$

(ii) $0 \leqslant \dfrac{f(s) - f(t)}{s - t} \leqslant L$

for all $s, t \geqslant 0$

if $0 < \lambda < \dfrac{\pi^2}{L} \quad \lambda \neq \dfrac{(90)^{1/2}}{L}$

then problem (1.1) has a unique solution in $H^2(0,1)$ given by an iterative sequence.

Proof. Problem (1.1) turns out to be equivalent to the following problem

$$(4.2) \begin{cases} \text{find } u \in L^2(0,1) \\ u(x) = \lambda \int_0^1 G(x,t) f(u(t)) dt \end{cases}$$

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where

$$G(x,t) = \begin{cases} (1-x)t, & 0 \leqslant t \leqslant x \\ x(1-t), & x \leqslant t \leqslant 1. \end{cases}$$

We make use of a functional approach to solve problem (4.2). Consider the Hilbert space

$$H = \{u \in L^2(0,1): u(0) = u(1) = 0\}.$$

Let us define the operator $T_\lambda: H \to H$ as

$$T_\lambda u(x) = \lambda \int_0^1 G(x,t) f(u,t))dt.$$

At first, remark that the following two point boundary value problem

$$(1.2) \quad \begin{cases} u'' + \lambda(au + b) = 0 \\ u(0) = u(1) = 0 \end{cases}$$

where $a\,b > 0$, function $f(s) = as + b$ satisfies (i) for $f(0) = b > 0$ and (ii) with $L = a$, has an unique solution if

$$0 \leqslant \lambda < \frac{\pi^2}{L}.$$

We claim that operator T_λ satisfies the hypotheses of Theorem 3.3:

(1) $(T_\lambda u - T_\lambda v, u - v) \geqslant 0$

(2) $\|T_\lambda u - T_\lambda v\| \leqslant \beta\|u - v\|$

where $\beta > 1$ for suitable λ.

Proof of ( ). Setting

$$\phi(t) = \frac{f(u(t)) - f(v(t))}{u(t) - v(t)}.$$

We have

$$\lambda \int_0^1 [u(x) - v(x)]dt \int_0^1 G(x,t)\phi(t)[u(t) - v(t)]dt$$

$$\geqslant \lambda \int_0^1 \int_0^1 G(x,t)\phi[u(t) - v(t)][u(x) - v(x)]dtdx$$

and therefore

$$(T_\lambda u - T_\lambda v, u - v) \geqslant 0.$$

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Proof of (2). We have

$$\|Tu(x) - Tv(x)\|^2 \leqslant \lambda^2 \int_0^1 \left(\int_0^1 G(x,t)[f(u(t) - f(v(t))]dt \right)^2 dx$$

$$\leqslant \lambda^2 L^2 \int_0^1 \left(\int_0^1 G^2(x\,t)dt \right) \left(\int_0^1 [u(t)-v(t)]^2 dt \right) dx$$

$$\leqslant \lambda^2 L^2 \|u - v\|^2 \int_0^1 \int_0^1 G^2(x,t)dtdx$$

$$= \lambda^2 L^2 K^2 \|u\ v\|^2,$$

where

$$K^2 = \int_0^1 \int_0^1 G^2(x,t)dtdx = 1/90$$

and thus we have

$$\|Tu - Tv\| \leqslant \beta \|u - v\|$$

where $\beta = \lambda LK$.

Using remark on two point boundary value problem (1.2) and Theorem 3.3 we deduce problem (1.1) having a unique solution when

$$\frac{1}{KL} < \lambda \leqslant \frac{\pi^2}{L},$$

given by the following iterative sequence:

$$\begin{cases} u_0(x) \in H \\ u_{n-1} = (1 - \mu)u_n + \mu\lambda \int_0^1 G(x,t) f(u_n(x))dt \end{cases}$$

for

$$0 \leqslant \mu \leqslant \frac{1}{\lambda LK - 1}.$$

On the other hand if

$$0 \leqslant \lambda \leqslant \frac{1}{KL}$$

operator T_λ turns out to be a contraction operator and hence sequence,

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$$\begin{cases} u_0(x) \in H \\ u_{n+1}(x) = T_\lambda u_n(x), \end{cases}$$

converges to the unique fixed point.

Corollary 4.3 *Assume f is a twice differentiable function such that for all $t \geqslant 0$*

(i) $f(t) > 0$

(ii) $0 \leqslant f'(t) \leqslant L$

(iii) $f''(t) < 0$

then for $0 < \lambda < \frac{\pi^2}{L}$ and $\lambda \neq \frac{(90)^{1/2}}{L}$ problem (1.1) has a unique solution given by an iteration process.

An example We ask for a solution to the following problem

$$\begin{cases} u'' + \lambda(u + log(1 + u) + 1) = 0 \\ u(0) = u(1) = 0. \end{cases}$$

It is immediate to show that the function $f(t) = t + log(1+t) + 1$ satisfies the hypotheses of Corollary 4.3 and that nonlinear two point boundary value problem has an unique solution for $0 < \lambda < \frac{\pi^2}{2}$ since one has $f'(t) = 1 + \frac{1}{1+t} \leqslant 2$. Such solutions are given by an iteration process for all

$$0 < \lambda < \frac{\pi^2}{2}, \quad \lambda \neq \left(\frac{45}{2}\right)^{1/2}.$$

Acknowledgement

The outhor is thankful to the MURST of Italy for partial support to this project.

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

FIXED POINTS FOR NONEXPANSIVE MULTIVALUED MAPPING AND THE OPIAL'S CONDITION

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ABSTRACT

Before Professor K L Singh's death some fixed point theorems for multivalued nonexpansive mappings on weakly compact and convex subsets of opial spaces, were discussed by both of us. In the present paper, these are reorganised.

INTRODUCTION

Several fixed point theorems have been obtained for nonexpansive mappings [1], [2], [3]. Fixed point theorems in lacally convex topological vector spaces for multivalued nonexpansive mappings have been studied [4], [10], [11], [12] and others. In this paper we shall continue these investigations.

Throughout the paper, we shall use E to denote a Hausdroff locally convex topological vector space, and F to denote the family of continuous seminorms generating the topology of E. Also $C(E)$ will denote the family of nonempty compact subsets of E.

For each $p \in F$ and $A, B \in C(E)$, we define

$\delta_p(A, B) = \sup\{p(a - b): a \in A \ b \in B\}$, and

$$D_p(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}[p(a-b)], \sup_{b\in B}\inf_{a\in A}[p(a-b)]\right\}.$$

Although p is only a seminorm, D_p is a Hausdroff metric on $C(E)$ [4].

Definition 1: Let K be a nonempty subset of E a mapping $T: K \to C(E)$ is called a multivalued contraction if there exists a constant k_p, $0 \leq k_p < 1$, such that for each $x, y \in K$, $D_p(Tx, Ty) \leq k_p p(x-y)$.

AMS (1980) *Subject Classification* Primary 47H10

*Died on November 22, 1990

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T is called multivalued nonexpansive if for each $x, y \in K$, $D_p(Tx, Ty) \leqq p(x-y)$.

Definition 2: A set C in a topological vector space E is called convex if $\alpha x + (1-\alpha) y \in C$ whenever $x, y \in C$ and $0 \leqq \alpha \leqq 1$.

For any $x, y \in E$, we set

$(x, y] = \{(1 - \alpha) x + \alpha y: 0 < \alpha \leqq 1\}$, and

$[x, y] = \{(1 - \alpha) x + \alpha y: 0 \leqslant \alpha \leqq 1\}$.

Definition 3: E is said to satisfy the Opial's condition if for each $x \in E$ and every net $\{x_\alpha\}$ converging weakly to x, then for each $p \in F$, we have

$$\lim p(x_\alpha - y) > \lim p(x_\alpha - x) \text{ for } y \neq x.$$

Definition 4: Let D be any nonempty subet of E, a mapping $T: D \to C(E)$ is said to satisfy the boundary condition (α) if for all $x \in D$ and all $y \in Tx$, $(x, y] \cap \neq D\phi$.

The following variant of Nadler's result [8] will be used.

Lemma. If $A, B \in C(E)$, then for each $a \in A$, there is a $b \in B$ such that $p(a - b) \leqq D_p(A, B)$ for all $p \in F$.

Theorem 1. Let K be a nonempty, weakly, compact and convex subset of E, and $T: K \to C(E)$ be nonexpansive. If E satisfies the Opial's condition, then $G_r(I - T)$ is closed in $E_w \times E$ where E_w is E with its weak topology and I is the identity mapping.

Proof: Let $\{(x_\alpha, y_\alpha)\}$ be a net in $G_r(I - T)$ such that $x_\alpha \to x$ weakly and $y_\alpha \to y$ It is not difficult to see that $x \in K$, so we must show that $y \in (I - T)(x)$. Now, for every α, there exists an $v_\alpha \in Tx_\alpha$ such that $y_\alpha = x_\alpha - v_\alpha$, by the Lemma, we can find an $v'_\alpha \in Tx$ such that $p(v_\alpha - v'_\alpha) \leqq D_p(Tx_\alpha, Tx)$. Thus we have

$$p(v_\alpha - v'_\alpha) \leqq D_r(Tx_\alpha, Tx) \leqq p(x_\alpha - x)$$

because T is nonexpansive. Therefore

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$$\lim p(x_\alpha - x) \geqq \lim p(v_\alpha - v'_\alpha) = \lim p(x_\alpha - y_\alpha - v'_\alpha).$$

By compactness of Tx, there is a convergent subnet in Tx still denoted by $\{v'_\alpha\}$ such that $v'_\alpha \to v$ for some $v \in Tx$. Therefore, we have

$$\lim p(x_\alpha - x) \geqq \lim p(x_\alpha - y - v).$$

Since E satisfies the Opial's condition, then we obtain $y + v = x$. That is $v = x - v \in (I - T)(x)$.

Theorem 2. Let E be an Opial's space, K a nonempty, weakly compact, and convex subset of E. $T: K \to C(E)$ be a multivalued non-expansive mapping which satisfies the boundary condition (α). Then T has a fixed point in K i e., there exists $x \in K$ such that $x \in Tx$.

Proof: We shall first show that $(I - T)(K)$ is closed.

Let y be a limit point of $(I - T)(K)$. Then there is a net $\{y_\alpha\}$ with $y_\alpha \in (I - T)x_\alpha$ for some $x_\alpha \in K$ and $y_\alpha \to y$. This implies that $x_\alpha - y_\alpha \in Tx_\alpha$ and $y_\alpha \to y$.

By the compactness of K, we know that there exists an $w \in K$ and a subnet $\{x_{\alpha_i}\}$ of $\{x_\alpha\}$ such that $x_{\alpha_i} \to w$ weakly. Therefore, there exists a subnet $\{y_{\alpha_i}\} \subseteq \{y_\alpha\}$ such that $y_{\alpha_i} \to y$, and by the Theorem 1, $y \in (I - T)(w)$. Thus, $(I - T)(K)$ is closed.

Let $z \in K$ be fixed and let $r_n = 1 - 1/n$ for each n. Define $T_n: K \to C(E)$ by $T_n(x) = r_n Tx + (1 - r_n)z$ for $x \in K$ Then $T_n x \in C(E)$ and T_n satisfies the boundary condition (α), since so does T. Each T_n is contraction map, it follows from a variant of Theorem 2 [6]. there exists an $x_n \in T_n x_n$. Therefore, $x_n \in r_n Tx_n + (1 - r_n)z$. This implies that there is $y_n \in Tx_n$, such that $x_n = r_n y_n + (1 - r_n)z$. Thus, $x_n - y_n = (r_n - 1)y_n + (1 - r_n)z \to 0$ as $n \to \infty$. Therefore, we shall have $0 \in (I-T)(K)$ because it is closed. Hence, there exists an $x \in K$ such that $x \in Tx$.

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Corollary. Let E be an Opial's space, K a nonempty, compact, and convex subset of E and $T:K\to C(E)$ be a multivalued nonexpansive mapping which satisfies the boundary condition (α). Then T has a fixed point in K.

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

A FINITE DOUBLE INTEGRAL INVOLVING A GENERAL MULTIVARIABLE POLYNOMIAL AND THE MULTIVARIABLE H-FUNCTION

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(Received: October 17, 1991; Revised: May 20, 1992)

ABSTRACT

In this paper we evaluate a general finite double integral involving the product of algebraic and exponential functions, a general multivariable polynomial, and the H function of several variables. Some new and interesiting special cases of our main integral formula have been considered briefly

1. INTRODUCTION

The general class of polynomials $S_n^m[x]$ introduced by Srivastva [3] has been further extended by Srivastava and Garg [4] to a multivariable polynomial in ihe following manner [4]:

$$S_n^{M_1,\ldots,M_r}[x_1, \ldots, x_r] = \sum_{\substack{M_1k_1+\ldots+M_rk_r \leqslant n \\ k_1,\ldots,k_r=0}} (-n)_{M_1k_1+\ldots+M_rk_r} \cdot A(n;k_1,\ldots,k_r) \frac{x_1^{k_1}}{k_1!} \ldots \frac{x_r^{k_r}}{k_r!}, \quad \ldots (1.1)$$

where $M_1, \ldots, M_r$ are arbitrary positive integers and the coefficients $A(n;k_1, \ldots, k_r)$ $(n\, k_i \geqslant 0,\ i = 1, \ldots, r)$ are arbitrary constants, real or complex.

The parameters of the multivariable H-function introduced and studied by Srivastava and Panda ([6], [7]) will be displayed in the following contracted notation [5, p. 251, Eq (C.1)].

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$$H[z_1,\ldots,z_r] = H^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix} (a_j;\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,p}: \\ b_j;\beta'_j,\ldots,\beta^{(r)}_j)_{1,q}: \end{matrix}\right.$$

$$\left.\begin{matrix} (c'_j,\gamma'_j)_{1,p_1};\ldots;(c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r} \\ (d'_j,\delta'_j)_{1,q_1};\ldots;(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r} \end{matrix}\right] \qquad \ldots (1.2)$$

Here, all the Greek letters are assumed to positive numbers for standardization purposes; the definition of the multivariable H-function given by (1.2) will, however, be meaningful even if some of these quantities are zero. The details of this function can be found in the papers and book referred to above.

We shall require the following known integrals ([1, p 450] and [2, p. 255]) for the evaluation of our main integral

$$\int_0^{\pi/2} e^{\omega(\alpha+\beta)\theta}\,(\sin\theta)^{\alpha-1}\,(\cos\theta)^{\beta-1}\,d\theta = \frac{e^{(\pi/2)\omega\alpha}\,\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \ldots (1.3)$$

$(Re(\alpha) > 0,\ Re(\beta) > 0,\ \omega = \sqrt{-1}\ ,$

$$\int_0^1 x^{c-1}(1-x)^{d-1}\,[ax+b(1-x)]^{-c-d}\,exp\,\{-zax/[ax+b(1-x)]\}$$

$$\cdot\ {}_2F_1\left[\alpha,\beta;c;\ \frac{ax}{ax+b(1-x)}\right]dx$$

$$= \frac{epx\{-z\}\ \Gamma(c)\,\Gamma(d)\,\Gamma(c+d-\alpha-\beta)}{a^c\,b^d\,\Gamma(c+d-\alpha)\,\Gamma(c+d-\beta)}\ {}_2F_2\left[d,c+d-\alpha-\beta\ ;\ c+d-\alpha,\ c+d-\beta\ ;\ z\right]$$

$$\ldots (1.4)$$

$Re(c) > 0,\ Re(d) > 0$; a and b are non-zero integers such that $ax + b(1-x) \neq 0,\ 0 \leqslant x \leqslant 1$).

2. MAIN INTEGRAL

We shall evaluate the following general integral :

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$$\int_0^1 \int_0^1 x^{c-1}\, y^{\rho-1}\,(1-x)^{d-1}\,(1-y)^{\sigma-1}\left\{(1-y^2)^{1/2}+\omega y\right\}^{\rho+2\sigma}\,[ax+b(1-x)]^{-c-d}$$

$$\cdot\, exp\left\{-zax/[ax+b(1-x)]\right\}\,{}_2F_1\left[\alpha,\beta;c;\frac{ax}{ax+b(1-x)}\right]$$

$$\cdot\, S_n^{M_1,\ldots M_r}\left[t_1 y^{u_1}(1-y^2)^{v_1}\left\{(1-y^2)^{1/2}+\omega y\right\}^{u_1+2v_1}\left\{\frac{1-x}{ax+b(1-x)}\right\}^{\delta_1},\ldots,\right.$$

$$\left. t_r y^{u_r}(1-y^2)^{v_r}\left\{(1-y^2)^{1/2}+\omega y\right\}^{u_1+2v_r}\left\{\frac{1-x}{ax+b(1-x)}\right\}^{\delta_r}\right]$$

$$\cdot\, H\left[z_1 y^{\rho_1}(1-y^2)^{\sigma_1}\left\{(1-y^2)^{1/2}+\omega y\right\}^{\rho_1+2\sigma_1}\left\{\frac{1-x}{ax+b(1-x)}\right\}^{\theta_1},\ldots,\right.$$

$$\left. z_s y^{\rho_s}(1-y^2)^{\sigma_s}\left\{(1-y^2)^{1/2}+\omega y\right\}^{\rho_s+2\sigma_s}\left\{\frac{1-x}{ax+b(1-x)}\right\}^{\theta_s}\right]dx\,dy$$

$$= \Gamma(c)\, exp\left\{-z+\pi\omega\rho/2\right\}\, a^{-c}\, b^{-d} \sum_{p=0}^{\infty} \sum_{k_1,\ldots,k_r=0}^{M_1k_1+\ldots+M_ik_r \leqslant n}$$

$$\cdot\, (-n)_{M_1k_1+\ldots+M_rk_r}\, A(n;k_1,\ldots,k_r)\,\frac{z^p}{p!}\prod_{i=1}^{r}\left\{\frac{t_i^{k_i}\,(b)^{-\delta_ik_i}\,exp\{\omega u_ik_i\pi/2\}}{k_i!}\right\}$$

$$\cdot\, H_{P+4\;\;Q+3}^{0,\;N+4:m_1.n_1\;;\ldots;m_s,n_s}\left\{\begin{matrix} z_1\;\; b^{-\theta_1}\; exp\;(\omega\pi\rho_1/2) \\ \vdots \\ z_s\;\; b^{-\theta_s}\; exp\;(\omega\pi\rho_s/2)\end{matrix}\right|$$

$$(1\;\; p-d-\sum_{i=1}^{r}\delta_ik_i\;;\;\theta_1,\ldots,\theta_s),\;(1-p\;\; d-c+\alpha+\beta-\sum_{i=1}^{r}\delta_ik_i\;;\;\theta_1,\ldots,\theta_s)$$

$$(b_j\;;\;\beta_j',\ldots,\beta_j^{(s)})_{1,Q}\;,\;(1-c-d\;\; p+\alpha-\sum_{i=1}^{r}\delta_ik_i\;;\;\theta_1,\ldots,\theta_s),$$

$$(1-\rho-\sum_{i=1}^{r}u_ik_i\;;\;\rho_1,\ldots,\rho_s),\;(1-2\sigma-2\sum_{i=1}^{r}v_ik_i\;;\;2\sigma_1,\ldots,2\sigma_s),$$

$$(1-c-d-p+\beta-\sum_{i=1}^{r}\delta_ik_i\;;\;\theta_1,\ldots,\theta_s),$$

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$$(a_j;\alpha'_j,\ldots,\alpha_j^{(s)})_{1,p} : (c'_j,\gamma'_j)_{1,p_1};\ldots;$$

$$(1-\rho-2\sigma-\sum_{i=1}^{r}(u_i+2v_i)k_i;\ \rho_1+2\sigma_1,\ldots,\rho_s+2\sigma_s):(d'_j,\delta'_j)_{1,q_1};\ldots;$$

$$\left.\begin{matrix}(c_j^{(s)},\gamma_j^{(s)})_{1,p_s}\\(d_s^{(s)},\delta_j^{(s)})_{1,q_s}\end{matrix}\right] \qquad \ldots (2.1)$$

The integral (2.1) is valid if the following sets of (sufficient) conditions are satisfied.

(i) $M_1,\ldots,M_r$ are arbitrary positive integers and the coefficients $A(n;k_1,\ldots,k_r)$ $(n;k_1,\ldots,k_r \geqslant 0)$ are arbitrary constants, real or complex.

(ii) a and b are non-zero integers such that $ax+b(1-x)\neq 0, 0\leqslant x\leqslant 1$.

(iii) $\min\limits_{1\leqslant i\leqslant r}\{Re(u_i),\ Re(v_i),\ Re(\delta_i)\} > 0.$

$\min\limits_{1\leqslant j\leqslant s}\{Re(\rho_j),\ Re(\sigma_j),\ Re(\theta_j)\} \geqslant 0.$

(iv) $Re(c) > 0,\ Re\ d) + \sum\limits_{i=1}^{s}\theta_i\xi_i > 0$, and

$$Re\ \rho) + \sum_{i=1}^{s}\rho_i\xi_i > 0,\ Re(\sigma) + \sum_{i=1}^{s}\sigma_i\xi_i > 0,$$

where $\xi_i = \min\limits_{1\leqslant j\leqslant m_i}\left\{Re(d_j^{(i)}/\delta_j^{(i)})\right\},\ i=1,\ldots,s$

(v) $U_i \geqslant 0,\ |arg\ z_i| < (1/2)\ U_i\ \pi\ (i=1,\ldots,s)$.

where

$$U_i = -\sum_{j=N+1}^{P}\alpha_j^{(i)} - \sum_{j=1}^{Q}\beta_j^{(i)} + \sum_{j=1}^{n_i}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i}\gamma_j^{(i)}$$

$$+ \sum_{j=1}^{m_i}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i}\delta_j^{(i)}$$

(vi) The series on the right-hand side of (2.1) is absolutely convergent.

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Proof. To establish the integral formula (2.1), we first use the series representation of the multivariable polynomial given by (1.1), and express the multivariable H-function in terms of its Mellin-Barnes contour integral. Then, interchanging the order of integration and summation (which is justified due to absolute convergence of the various integrals involved in the proces under the conditions stated with result) and evaluating the x- and y-integrals with the help of (1.3) (1.4), respectively, we find that

$$L.\ H.\ S.\ \text{of}\ (2.1) = \sum_{k_k,\ldots,k_r=0}^{M_1k_1+\ldots+M_rk_r\leqslant n} (-n)_{M_1k_1+\ldots+M_rk_r}$$

$$\cdot\ A(n;k_1,\ldots,k_r)\ \prod_{i=1}^{r}\left\{\frac{t_i^{k_i}}{k_i!}\right\}\ \frac{1}{(2\pi\omega)^s}\int_{L_1}\cdots\int_{L_s}\phi(\xi_1,\ldots,\xi_s)$$

$$\prod_{j=1}^{s}\left\{\theta_j(\xi_j)\,z_j^{\xi_j}\right\}\left(\frac{exp\{-z\}\ \Gamma c)\,\Gamma(d+\sum_{i=1}^{r}\delta_ik_i+\sum_{j=1}^{s}\theta_j\xi_j)}{(a)^c\,(b)^{d+\sum_{i=1}^{r}\delta_ik_i+\sum_{j=1}^{s}\theta_j\xi_j}}\right.$$

$$\left.\frac{\Gamma(c+d\ \alpha-\beta+\sum_{i=1}^{r}\delta_ik_i+\sum_{j=1}^{s}\theta_j\xi_j)}{\Gamma(c+d-\alpha+\sum_{i=1}^{r}\delta_ik_i+\sum_{j=1}^{s}\theta_j\xi_j)\,\Gamma(c+d-\beta+\sum_{i=1}^{r}\delta_ik_i+\sum_{j=1}^{s}\theta_j\xi_j)}\right.$$

$$\cdot\ {}_2F_2\left[d+\sum_{i=1}^{r}\delta_ik_i+\sum_{j=1}^{s}\theta_j\xi_j,\ c+d\ \alpha-\beta+\sum_{i=1}^{r}\delta_ik_i+\sum_{j=1}^{s}\theta_j\xi_j\ ;\right.$$

$$\left.\left.c+d-\alpha+\sum_{i=1}^{r}\delta_ik_i+\sum_{j=1}^{s}\theta_j\xi_j\ \ c+d-\beta-\sum_{i=1}^{r}\delta_ik_i+\sum_{j=1}^{s}\theta_j\xi_j\ ;\ z\right]\right)$$

$$\cdot\left(exp\left\{\rho+\sum_{i=1}^{r}u_ik_i+\sum_{j=1}^{s}\rho_j\xi_j\right\}\Gamma(\rho+\sum_{i=1}^{r}u_ik_i+\sum_{j=1}^{s}\rho_j\xi_j)\right.$$

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$$\cdot \frac{\Gamma(2\sigma + 2\sum_{i=1}^{r} \nu_i k_i + 2\sum_{j=1}^{s} \sigma_j \xi_j)}{\Gamma(\rho + 2\sigma + \sum_{i=1}^{r} (u_i + 2\nu_i)k_i + \sum_{j=1}^{s} (\rho_j + 2\sigma_j)\xi_j)} \Bigg) d\xi_1, \ldots, d\xi_s. \quad \ldots(2.2)$$

On expressing ${}_2F_2[x]$ as a hypergeometric series and changing the oteder of integration and summation, if we interpret the resulting contour integral in terms of the multivariable H-function, we get at once the integral (2.1).

3. SPECIAL GASES

I. Taking $u_i = \nu_i = 0\ (i=1,\ldots,r)$, $\rho_j = \sigma_j = 0\ (j=1,\ldots,s)$ in (2.1), and evaluating the y-integral, we arrive at the following integral which is believed to be new.

$$\int_0^1 x^{\sigma-1}(1-x)^{d-1}\,[ax+b(1-x)]^{-\sigma-d}\,exp\left\{-zax/[ax+b(1-x)]\right\}$$

$$\cdot\ {}_2F_1\left[\alpha,\beta,c;\frac{ax}{ax+b(1-x)}\right] S_{\cdot}^{M_1,\ldots,M_r}\left[t_1\left\{\frac{1-x}{ax+b(1-x)}\right\}^{\delta_1},\ldots,\right.$$

$$\left. t_r\left\{\frac{1-x}{ax+b(1-x)}\right\}^{\delta_r} H\left[z_1\left\{\frac{1-x}{ax+b(1-x)}\right\}^{\theta_1},\ldots,z_s\left\{\frac{1-x}{ax+b(1-x)}\right\}^{\theta_s}\right]\right] dx$$

$$= \Gamma(c)\,e^{-z}\,a^{-\sigma}\,b^{-d} \sum_{p=0}^{\infty} \sum_{k_1,\ldots,k_r=0}^{M_1k_1+\ldots+M_rk_r \leqslant n} (-n)_{M_1k_1+\ldots+M_rk_r}$$

$$\cdot\ A(n;k_1,\ldots k_r)\,\frac{z^p}{p!}\prod_{i=1}^{r}\left\{\frac{t_i^{k_i}\,(b)^{\delta_i k_i}}{k_i!}\right\}$$

$$\cdot\ H^{0}_{P+2,Q+2:\,p_1,q_1;\ldots;p_s,q_s}\left[\begin{matrix} z_1\,b^{-\theta_1} \\ \vdots \\ z_s\,b^{-\theta_s}\end{matrix}\right| \begin{matrix}(1-p-d-\sum_{i=1}^{r}\delta_i k_i;\theta_1,\ldots,\theta_s), \\ (b_j;\beta_j',\ldots,\beta_j^{(s)})_{1,Q}\end{matrix}\ ,$$

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$$(1-p-c-d+\alpha+\beta-\sum_{i=1}^{r}\delta_i k_i;\ \theta_1,\ldots,\theta_s),\ (a_j;\alpha'_j,\ldots,\alpha_j^{(s)})_{1,p}\ :$$

$$(1-p-c-d+\alpha-\sum_{i=1}^{r}\delta_i k_i;\ \theta_1,\ldots,\theta_s),(1-p-c-d+\beta-\sum_{i=1}^{r}\delta_i k_i;\ \theta_1,\ldots,\theta_s)\ :$$

$$\left.\begin{matrix}(c'_j,\gamma'_j)_{1,p_1};\ldots;(c_j^{(s)},\gamma_j^{(s)})_{1,p_s}\\ (d'_j,\delta'_j)_{1,q_1};\ldots;(d_j^{(s)},\delta_j^{(s)})_{1,q}\end{matrix}\right], \qquad \ldots (3.1)$$

provided that

(i) $Re(\delta_i) \geqslant 0\ (i = 1,\ldots,r),\ Re\,\theta_j) \geqslant 0,\ j = 1,\ldots,s;$

(ii) $Re(c) > 0,\ Re(d) + \sum_{j=1}^{s}\theta_j\xi_j > 0,$

where $\xi_j = \min_{j\leqslant i\leqslant m_j}\left\{Re(d_i^{(j)}/\delta_i^{(j)})\right\},\ j = 1,\ldots,s;$

(iii) conditions (i), (ii), (v) and (vi) stated with (2.1) are also satisfied.

II. Similarly, if we put $\delta_i = 0\ (i = 1,\ldots,r)$ and $\theta_j = (j = 1,\ldots,s)$ in (2.1), we get the following result:

$$\int_0^1 y^{\rho-1}(1-y^2)^{\sigma-1}\left((1-y^2)^{1/2}+\omega y\right)^{\rho+2\sigma} S_n^{M_1,\ldots M_r}\Big[t_1\, y^{u_1}(1-y^2)^{v_1}$$

$$\left\{(1-y^2)^{1/2}+\omega y\right\}^{u_1+2v_1},\ldots,t_r\, y^{u_r}(1-y^2)^{v_r}\left\{(1-y^2)^{1/2}+\omega y\right\}^{u_r+2v_r}\Big]$$

$$\cdot\ H\Big[z_1\, y^{\rho_1}(1-y^2)^{\sigma_1}\left\{(1-y^2)+\omega y\right\}^{\rho_1+2\sigma_1},\ldots,z_s\, y^{\rho_s}(1-y^2)^{\sigma_s}$$

$$\cdot\ \left\{(1-y^2)^{1/2}+\omega y\right\}^{\rho_s+2\sigma_s}\Big]\,dy$$

$$= \exp\{\omega\rho\pi/2\}\sum_{\substack{M_1k_1+\ldots+M_rk_r\leqslant n\\ k_1,\ldots,k_r=0}}(-n)_{M_1k_1+\ldots+M_rk_r}$$

$$\cdot\ A(n;k_1,\ldots,k_r)\prod_{i=1}^{r}\left\{\frac{t_i^{k_i}\exp\{\omega u_i k_i\pi/2\}}{k!}\right\}$$

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$$\cdot\ H^{0,N+2:\ m_1,n_1;\ldots;m_s,n_s}_{P+2,Q+1:\ p_1,q_1;\ldots;p_s,p_s}\left[\begin{matrix} z_1\ exp\ \{\omega\ \rho_1\pi/2\} \\ \vdots \\ z_s\ exp\ \{\omega\ \rho_s\pi/2\}\end{matrix}\right|$$

$$(1-\rho-\sum_{i=1}^{r} u_i k_i;\rho_1,\ldots,\rho_s),\ (1-2\sigma-2\sum_{i=1}^{r} v_i k_i;\ 2\sigma_1,\ldots,2\sigma_s).$$

$$(b_j;\beta'_j,\ \ldots\ ,\beta^{(s)}_j)_{1,Q}\ ,\ (1-\rho-2\sigma-\sum_{i=1}^{r}(u_i+2v_i)k_i;\ \rho_1+2\sigma_1,\ldots,\rho_s+2\sigma_s),$$

$$\left.\begin{matrix}(a_j;\alpha'_j,\ldots,\alpha^{(s)}_j)_{1,P} & :(c'_j,\gamma'_j)_{1,p_1};\ldots;(c^{(s)}_j,\gamma^{(s)}_j)_{1,p_s} \\ & :(d',\delta'_j)_{1,q_1};\ldots;(d^{(s)}_j,\delta^{(s)}_j)_{1,p_s}\end{matrix}\right] \qquad \ldots (3.2)$$

The equation (3.2) is valid if

(i) $min\ \{Re(u_i), Re(v_i)\} \geqslant 0\ (i = 1,\ldots,r);$
$min\ \{Re(\rho_j), Re(\sigma_j)\} \geqslant 0\ (j = 1,\ldots,s);$

(ii) $Re(\rho) + \sum_{i=1}^{s} \rho_i \xi_i \geqslant 0$ and $Re(\sigma) + \sum_{i=1}^{s} \sigma_i \xi_i > 0$,

where $\xi_i = \min_{1\leqslant j\leqslant m_i} \left\{ Re\ d^{(i)}_j/\delta^{(i)}_j \right\}\ (i = 1,\ldots,s),$

(iii) Conditions (i), (v) and (vi) stated with (2.1) are also satisfied.

III. If we take $M_i = 0\ (i = 2,\ldots r),\ t_i = 1$, and replace $A(n;k_1,\ldots,k_r)$ by $A\ n;k)$ therein, we easily arrive at the following general integral:

$$\int_0^1\int_0^1 x^{\sigma-1}\ y^{\rho-1}\ (1-x)^{d-1}\ (1-y^2)^{\sigma-1}\ \{(1-y^2)^{1/2}+\omega y\}^{\rho+2\sigma}$$

$$[ax+b(1-x]^{-\sigma-d}\ exp\ \{\ zax/\ [ax+b(1-x)]\ \}\ {}_2F_1\left[\alpha\ \beta;c;\frac{ax}{ax+b(1-x)}\right]$$

$$\cdot\ S^M_N\left[t\ y^u\ (1-y^2)^v\ \{(1-y^2)^{1/2}+\omega y\}^{u+2v}\ \left\{\frac{1-x}{ax+b(1-x)}\right\}^{\delta}\right]$$

$$\cdot\ H\Bigg[\ z_1\ y^{\rho_1}\ (1-y^2)^{\sigma_1}\left\{(1-y^2)^{1/2}+\omega y\right\}^{\rho_1+2\sigma_1}\left\{\frac{1-x}{ax+b(1-x)}\right\}^{\theta_1},\ldots,$$

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$$z_s y^{\rho_s} (1-y^2)^{\sigma_s} \{(1-y^2)^{1/2}+\omega y\}^{\rho_s+2\sigma_s} \left\{\frac{1-x}{ax+b(1-x)}\right\}^{\theta_s}\Big] dx\, dy$$

$$= \Gamma(c)\, exp\ \{-z+\omega(\pi/2)\rho\}\ a^{-c}\, b^{-d}$$

$$\cdot \sum_{p=0}^{\infty} \sum_{k=0}^{[n/m]} (-n)_{mk}\, A(n;k)\, \frac{t^k}{k!}\, b^{\delta k}\, exp\ \{uk\rho\,\pi/2\}\ \frac{z^p}{p!}$$

$$\cdot H^{0,N+4:m_1,n_1;\cdots;m_s,n_s}_{P+4,Q+3:p_1,q_1;\cdots;p_s,q_s}\left[\begin{matrix} z_1 b^{-\theta_1}\ exp\ \{\omega\rho_1\pi/2\} \\ \vdots \\ z_s b^{\theta_s}\ exp\ \{\omega\rho_s\pi/2\}\end{matrix}\right|$$

$(1-p-d-\delta k;\ \theta_1,\cdots,\theta_s),\ (1-p-c-d+\alpha+\beta-\delta k;\ \theta_1,\cdots,\theta_s),$

$(b_j;\beta'_j,\cdots,\beta^{(s)}_j)_{1,\theta},\ (1-c-d-p+\alpha-\delta k;\ \theta_1,\cdots,\theta_s),$

$(1-\rho-uk;\ \rho_1,\cdots,\rho_s),\ (1-2\sigma-2vk;\ 2\sigma_1,\cdots,2\sigma_s),$
$(1-c-d-p+\beta-\delta k;\ \theta_1,\cdots,\theta_s),$

$(a_j;\alpha'_j,\cdots,\alpha^{(s)}_j)_{1,P}$ $:(c'_j,\gamma'_j)_{1,p_1};\cdots;$

$1-\rho-2\sigma-(u+2v)\,k;\rho_1+2\sigma_1,\cdots,\rho_s+2\sigma_s)$ $:(d'_j,\delta'_j)_{1,q_1};\cdots;$

$$\left.\begin{matrix}(c^{(s)}_j,\gamma^{(s)}_j)_{1,p_s} \\ (d^{(s)}_j,\delta^{(s)}_j)_{1,q_s}\end{matrix}\right] \qquad \ldots (3.3)$$

provided that the conditions easily obtainable from those mentioned with the main integral (2.1) are satisfied.

The integrals (3.1), (3.2) and (3.3) are also quite general in nature. By suitably specializing the arbitrary coefficients in the general class of polynomials and the parameters of the multivariable *H*-function, a number of (known and new) integrals can be evaluated.

Acknowledgements

The author is thankful to Dr. S. P. Goyal (Department of Mathematics, University of Rajasthan, Jaipur) for his help and guidance during the preparation of this paper. I am also thankful to Professor H. M. Srivastava (University of Victoria, Canada) for the veryy careful reading of the manuscript and for many constructive suggestions.

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

ON THE INVERSE LAPLCE TRANSFORM OF CERTAIN FUNCTIONS INVOLVING ONE AND MORE VARIABLES

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(Received: November 12, 1991; Revised: May 20, 1992)

ABSTRACT

In this paper, we first establish three interesting and useful formulas giving the inverse Laplace transform of various products of algebraic powers and general classes of polynomials involving one and more variables. The generalizations of these formulas have also been established in § 3. In view of the generality of the polynomials considered here, on specializing the coefficients, our results would reduce to a large number of formulas giving inverse Laplace transforms of simpler functions and known classes of polynomials.

1. RESULTS REQUIRED

$$L^{-1}\left\{(p^2+b^2)^{-(2\mu+1)/2}\, S_N^M\left[z(p^2+b^2)^{-\eta}\right]; x\right\} \cdots$$

$$= \sqrt{\pi}\left(\frac{x}{2b}\right)^{\mu} \sum_{k=0}^{[N/M[} (-N)_{MK}\, B_{N,K}\, \frac{z^k}{k!}\, \frac{1}{\Gamma(\mu+\eta k+(1/2))}\left(\frac{x}{2b}\right)^{\eta k} J_{\mu+\eta k}(bx) \qquad \ldots (1.1)$$

where

$$S_N^M[x] = \sum_{k=0}^{[N/M]} (-N)_{MK}\, B_{N,K}\, \frac{z^k}{k!} \qquad \ldots (1.2)$$

stands for a generol class of polynomials introduced by Srivastava ([4, p. 1, Eq. (1)]; see also [7]). The result (1.1) is valid under the following conditons:

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$\eta \geqslant 0$, $Re(p) > 0$ and $Re(2\mu+1) > 0$.

$$L^{-1}\left\{\{p+(p^2+b^2)^{1/2}\}^{-\nu} S_N^M\left[z\{p+(p^2+b^2)^{1/2}\}^{-\rho}\right]; x\right\}$$

$$= b^{-\nu} \sum_{k=0}^{[N/M]} (-N)_{MK} B_{N,K} \frac{z^k}{k!} (\nu+\rho k) b^{\rho k} x^{-1} J_{\nu+\rho k}(dx) \quad \ldots (1.3)$$

provided that

$$\rho \geqslant 0, \; Re(p) > 0.$$

$$L^{-1}\left\{p^{-2\lambda}(p^2+a^2)^{-\nu} S_n^{m_1,\ldots,m_s}\left[y_1 p^{-2\rho_1}(p^2+a^2)^{-\sigma_1}, \ldots, y_s p^{-2\rho_s}(p^2+a^2)^{-\sigma_s}\right]; x\right\}$$

$$= \sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s \leqslant n} (-n)_{m_1k_1+\ldots+m_sk_s} A(n;k_1,\ldots,k_s) \frac{y_1^{k_1} \ldots y_s^{k_s}}{k_1!\ldots k_s!} \cdot \frac{x^{2\lambda'+2\nu'-1}}{\Gamma(2\lambda'+2\nu')} {}_1F_2\left[\nu';\lambda'+\nu',\lambda'+\nu'+1/2; -\frac{a^2x^2}{4}\right] \quad \ldots (1.4)$$

where

$$\lambda' = \lambda + \sum_{i=1}^{s} \rho_i k_i \text{ and } \nu' = \nu + \sum_{i=1}^{s} \sigma_i k_i \quad \ldots (1.5)$$

and $\rho_i, \sigma_i \geqslant 0$ $(i=1,\ldots,s)$, $Re(p) > 0$, $Re(\lambda) > 0$, $Re(\nu) \geqslant 0$.

Also

$$S_n^{m_1,\ldots,m_s}[y_1,\ldots,y_s] = \sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s \leqslant n} (-n)_{m_1k_1+\ldots m_sk_s} \cdot A(n;k_1,\ldots,k_s) \frac{y_1^{k_1}}{k_1!} \ldots \frac{y_s^{k_s}}{k_s!} \quad \ldots (1.6)$$

stands for a general class of multivariable polynomials (see, e.g., [5]).

$$L^{-1}\left\{(p^2+a^2)^{1/2}\{p+(p^2+a^2)^{1/2}\}^{-\lambda} S_n^{m_1,\ldots m_s}\right.$$

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$$\left[y_1\ \{p+(p^2+a^2)^{1/2}\}^{-\rho_1},\ldots,y_s\{p+(p^2+a^2)^{1/2}\}^{-\rho_s}\right]\ ;\ x\ \Big\}$$

$$= \sum_{\substack{m_1k_1+\ldots+m_sk_s \leqslant n \\ k_1,\ldots,k_s=0}} (-n)_{m_1k_1+\ldots+m_sk_s}\ A(n;k_1,\ldots,k_s)$$

$$\cdot \frac{y_1^{k_1}\ldots y_s^{k_s}}{k_1!\ldots k_s!}\, a^{-\lambda'}\ J_{\lambda'}(ax) \qquad \ldots(1.7)$$

where λ' is given by (1.5) and

$\rho_i > 0,\ i = 1,\ldots,s\ ;\ Re(\lambda) > -1,\ Re(p) > 0.$

$$L^{-1}\ \left\{p^{-\nu}\,q^{-\mu}\,S_N^M\left[z\,p^{-\rho}\,q^{-\sigma}\right]\ ;\ x,y\ \right\}$$

$$= x^{\nu-1}\,y^{\mu-1} \sum_{k=0}^{[N/M]} (-N)_{MK}\ B_{N,K}\ \frac{z^K\ x^{\rho K}\ y^{\sigma K}}{k!\ \Gamma(\nu+\rho k)\ \Gamma(\mu+\sigma k)} \qquad \ldots(1.8)$$

where

$\rho,\ \sigma \geqslant 0,\ Re(\nu) > 0,\ Re(\mu) > 0,\ Re(p) > 0,\ Re(q) > 0,$

he results (1.1), (1.3), (1.4), (1.7) and (1.8) are not available in the literature. In order to evaluate (1.1) use series representation for $S_N^M[x]$ given by (1.2) and make use of a known result ([2, *p.*239, *Eq.* (18)]). Similarly the results (1.3), (1.4), (1.7) and (1.8) can be established if we use the known results ([2, *p.* 240, *Eq.* (23)]), ([2, *p.* 238, *Eq* (10)]), ([2, *p.* 240, *Eq.* (21)]), ([1, *p.* 137, *Eq.* (287)]) respectively instead of ([2, *p.* 239, *Eq* (18)]).

$$L^{-1}\ \left\{\ p^{ms_2}\ q^{ns_1}\ (p^mq^n+a)^{-s_1-s_2}\ ;\ x,\ y\ \right\}$$

$$= \frac{x^{ms_1-1}\ \ y^{ns_2-1}}{\Gamma(s_1+s_2)}\ {}_1\Psi_2\left[\begin{matrix}(s_1+s_2,\ 1)\\ (ms_1,m),\ (ns_2,n)\end{matrix}\ ;\ -ax^my^n\right] \qquad \ldots(1.9)$$

$Re(s_1) > 0,\ Re(s_2) > 0,\ Re(p) > 0,\ Re(q) > 0,\ m,n > 0.$

The result (1.9) is a special case of a known result ([3, *p.* 118] ; see also [6, *p.* 69, *Eq.* (5. 4. 1)]).

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1. MAIN RESULTS

$$L^{-1}\left\{p^{-2\lambda}\ (p^2+a^2)^{-\nu}\ (p^2+b^2)^{-(2\mu+1)/2}\ S_N^M\left[z(p^2+b^2)^{-\eta}\right]\right.$$

$$\left..S^{m_1,\ldots,m_s}\left[y_1\ p^{-2\rho_1}\ (p^2+a^2)^{-\sigma_1},\ldots,y_s\ p^{-2\rho_s}\ (p^2+a^2)^{-\sigma_s}\right];x\right\}$$

$$=\sum_{K=0}^{[N/M]}\ \sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s\leqslant n}(-n)_{m_1k_1+\ldots+m_sk_s}$$

$$(-N)_{MK}\ B_{N,K}\ A(n;k_1,\ldots,k_s)\ .\ \frac{y_1^{k_1}\ldots y_s^{k_s}\ z^K(x)}{k_1!\ldots k_s!\ k!\ \Gamma(1+2\lambda'+2\nu'+2\mu+2\eta K)}\ \frac{2\lambda'+2\nu'+2\mu+2\eta k}{}$$

$$.\ F\begin{matrix}0:1;1\\2:0;0\end{matrix}\left[\begin{matrix}--\\1/2+\lambda'+\nu'+\mu+\eta K,\ 1+\lambda'+\nu'+\mu+\eta K\end{matrix}:\right.$$

$$\left.\begin{matrix}\nu\ ;\ \mu+\eta K+(1/2)\ ;\\ --\ ;\qquad --\ ;\end{matrix}\ \frac{-a^2x^2}{4},\ \frac{-b^2x^2}{4}\right]\qquad\ldots(2.1)$$

where λ', ν' are given by (1.5), the conditions of validity of (2.1) are

(i) $\rho_i,\ \sigma_i,\ \eta_i \geqslant 0$, where $i=1,\ldots s$,

$Re(p) > 0,\ Re(\lambda+\nu) > 0,\ Re(2\mu+1) > 0$

(ii) $M,\ m_1,\ldots,m_s$ are arbitrary positive integers, the coefficients $B_{N,K},\ A(n;k_1,\ldots,k_s),\ (N, n, K, k_1,\ldots,k_s \geqslant 0)$ are arbitrary constants, real or complex.

$$L^{-1}\left\{(p^2+b^2)^{1/2}\ \{p+(p^2+a^2)^{1/2}\}^{-\lambda}\ \{p+(p^2+b^2)^{1/2}\}^{-\nu}\right.$$

$$\left.S_N^M\left[z\{p+(p^2+b^2)^{1/2}\}^{-\rho}\right]\right.$$

$$\left.,S_n^{m_1,\ldots,m_s}\left[y_1\{p+(p^2+a^2)^{1/2}\}^{-\rho_1},\ldots,y_s\ \{p+(p^2+a^2)^{1/2}\}^{-\rho_s}\right];x\right\}$$

$$=\sum_{K=0}^{[M/N]}\ \sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s\leqslant n}(-n)_{m_1k_1+\ldots+m_sk_s}\ (-N)_{MK}\ B_{N,K}$$

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$$A(n;k_1,\ldots,k_s)\ \frac{z^K}{K!}\ \frac{y_1^{k_1}}{k_1!}\cdots\frac{y_1^{k_s}}{k_s!}\ \frac{(x/2)^{\lambda'+\nu+\rho K}}{\Gamma(\lambda'+\nu+\rho K)}$$

$$F^{0:2;2}_{2:1;1}\left[\begin{matrix} -- & : \tfrac{1}{2}(\lambda'+2),\ \tfrac{1}{2}(\lambda'+1)\ ; \\ \tfrac{1}{2}(\lambda'+\nu+\rho K+2),\ \tfrac{1}{2}(\lambda'+\nu+\rho K+1) & :\ \lambda'+1\ ; \end{matrix}\right.$$

$$\left.\begin{matrix} \tfrac{1}{2}(\nu+\rho K),\ \tfrac{1}{2}(\nu+\rho K+1)\ ; \\ \nu+\rho K+1\ ; \end{matrix}\ \frac{-a^2x^2}{4},\ \frac{-b^2x^2}{4}\right] \qquad \ldots(2.2)$$

provided that

(i) $\rho,\ \rho_i \geqslant 0\ (i=1,\ldots,s)$, $Re(\lambda)$, $Re(\nu)$ and $Re(p) > 0$

(ii) $M,\ m_1,\ldots,m_s$ are arbitrary positive integers. $B_{N,K}$. $A(n;k_1,\ldots,k_s)$ are arbitrary constants, real or complex.

Also λ' is defined by (1.5)

$$L^{-1}\left\{ p^{ms_2-\nu}\ q^{ns_1-\mu}\ (p^m q^n+a)^{-s_1-s_2}\ S_N^M\left[z\,p^{-\rho}\ q^{-\sigma}\right];x,y\right\}$$

$$=\frac{x^{ms_1+\nu-1}\ y^{ns_2+\mu-1}}{\Gamma(s_1+s_2)}\sum_{K=0}^{[N/M]}(-N)_{MK}\,B_{N,K}\ \frac{z^K}{K!}\ x^{-\rho K}\ y^{-\sigma K}$$

$$\cdot\ {}_1\Psi_2\left[\begin{matrix}(s_1+s_2,\ 1) \\ (\nu+\rho K+ms_1,m),\ (\mu+\sigma K+ns_2,n)\end{matrix}\ ;-ax^m\ y^n\right] \qquad \ldots(2.3)$$

provided that

(i) $\rho,\ \sigma,\ Re(\nu),\ Re(\mu),\ Re(p),\ Re(q) > 0$, $m\ n$ are real numbers.

(ii) M is arbitrary positive integer and $B_{N,K}$ $(N,\ K \geqslant 0)$ is an arbitrary constant, real or complex.

Proof of (2.1). Using the well-known convolution theorem for Laplace transforms to the pair of equations (1.1) and (1.4), we find that

$$L.H.S.\ \text{of}\ (2.1)=\sum_{K=0}^{[N/M]}\ \sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s\leqslant n}(-n)_{m_1k_1+\ldots+m_sk_s}$$

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$$(-N)_{MK}\, A(n;k_1,\ldots,k_s)\, B_{N\,K}\, \frac{\sqrt{\pi}\ (2b)^{-\mu-\eta K}}{(\Gamma(\mu+\eta K+1/2))\ \Gamma(\lambda'+2\nu')}\, \frac{z^K}{K!}\, \frac{y_1^{k_1}}{k_1!}\cdots\frac{y_s^{k_s}}{k_s!}$$

$$\int_0^x (x-u)^{\mu+\eta K}\, J_{\mu+\eta K}\ \{b(x-u)\}\ u^{2\lambda'+2\nu'-1}$$

$$ {}_1F_2\left[\nu';\ \lambda'+\nu',\ \lambda'+\nu'+\tfrac{1}{2};\ \frac{-a^2u^2}{4}\right] du \qquad \ldots(2.4)$$

Now, using the known result ([6, *p.* 61, *Eq.* (5.2.1)]) to evaluate u-integral and the results ([6, *p.* 15. *Eq.* (2. 3. 6); *p.* 19, *Eq.* (2.6.11)]) and the Gamma duplication formula to simplify the expression so obtained, we easily arrive at the required result (2.1).

Proof of (2.2). To prove the result (2.2), we proceed on the lines similar to those in the proof of (2.1). Here we use the transform pairs (1.3) and (1.7), instead of (1.1) and (1.4).

Poorf of (2 3) To prove (2.3), we apply the convolution theorem of Laplace transforms for two variables for the pair (1.8) and (1.9), and we get

$$L.H.S.\ \text{of}\ (2.3) = \frac{1}{\Gamma(s_1+s_2)} \sum_{K=0}^{[N/M]} (-N)_{MK}\, B_{N,K}\, \frac{z^K}{K!}\, \frac{1}{\Gamma(\nu+\rho K)\Gamma(\mu+\sigma K)}$$

$$\cdot \int_0^x \int_0^y (x-u)^{\nu+\rho K-1}\ (y-u)^{\mu+\sigma K-1}\ u^{ms_1-1}\ v^{ns_2-1}$$

$$\cdot\ {}_1\Psi_2\left[\begin{matrix}(s_1+s_2,\ 1)\\ (ms_1,\ m),\ (ns_2,\ n)\end{matrix};\ -au^m v^n\right] du\, dv \qquad \ldots(2.5)$$

Writing ${}_1\Psi_2$ in terms of its bilateral series, changing the order of integration and summation, and evaluating the u, v-integrals, we easily arrive at the desired result (2.3).

3. GENERALIZATIONS OF THE FORMULAS (2.1) TO (2.3)

Now we establish three more formulas (given below), which are generalizations of the formulas (2.1) to (2.3).

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$$L^{-1}\left\{p^{-2\lambda}(p^2+a^2)^{-\nu}\, S_n^{m_1,\cdots,m_s}\left[y_1 p^{-2\rho_1}(p^2+a^2)^{-\sigma_1} ,\cdots, y_s p^{-2\rho_s}(p^2+a^2)^{-\sigma_s}\right]\right.$$

$$\left.\cdot \prod_{i=1}^{r}\left\{ (p^2+b_i^2)^{-(2\mu_i+1)/2}\; S_{N_i}^{M_i}\left[z_i(p^2+b_i^2)^{-\eta_i}\right]\right\};\, x\right\}$$

$$= x^{2\lambda+2\nu+2\sum_{i=1}^{r}(\mu_i)+r-1}$$

$$\sum_{k_1,\cdots,k_s=0}^{m_1k_1+\cdots+m_sk_s\leqslant n}\ \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}(-n)_{m_1k_1+\cdots+m_sk_s}$$

$$A(n;k_1,\cdots,k_s)\ \frac{y_1^{k_1}}{k_1!}\cdots\frac{y_s^{k_s}}{k_s!}\ \sum_{i=1}^{r}\left\{(-N_i)_{M_iK_i}\ B_{N_i,K_i}\ \frac{(z_i)^{K_i}}{K_i!}\right\}$$

$$\cdot\ \frac{\Gamma^{(r)}\left(2\lambda+2\nu+r+2\sum_{j=1}^{s}(\rho_j+\sigma_j)k_j+2\sum_{i=1}^{r}(\mu_i+\eta_iK_i)\right)}{\Gamma\left(2\lambda+2\nu+r+2\sum_{j=1}^{s}(\rho_j+\sigma_j)k_j+2\sum_{i=1}^{r}(\mu_i+\eta_iK_i)\right)}$$

$$F^{0:1;\ldots;1}_{2:0;\ \ ;0}\left[\begin{matrix}---\\ \lambda+\nu+\sum_{j=1}^{s}(\rho_j+\sigma)k_j+\sum_{i=1}^{r}\mu_i+\eta_iK_i)+\frac{r}{2}\end{matrix}\right.$$

$$\begin{matrix}---\\ \lambda+\nu+\sum_{j=1}^{s}(\rho_j+\sigma_j)k_j+\sum_{i=1}^{r}(\mu_i+\eta_iK_i)+\frac{r+1}{2}\end{matrix}:\ \begin{matrix}---\\ \mu_r+\eta_rK_r+\frac{1}{2}\end{matrix};$$

$$:\ \nu+\sum_{j=1}^{s}\rho_jk_j:\mu_1+\eta_1K_1+;\ldots;$$

$$:\ \ --\ \ :\ \ --$$

$$\left.--\ \ ;\ \ \frac{-a^2x^2}{4},\frac{-b_1^2x^2}{4},\ ,\frac{-b_r^2x^2}{4}\right]. \qquad \ldots(3.1)$$

provided that

(i) $Re(p)\geqslant 0$, $Re(\lambda+\nu)\geqslant 0$, $Re(2\mu+1)\geqslant 0$. ρ_j, σ_j, $\geqslant 0$, and

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$\eta_i,\ \mu_i > 0\ (i=1,\ldots,r,\ j=1,\ldots,s)$

(ii) $m_1,\ldots,m_s,\ M_1,\ \ldots,M_r$ are arbitrary positive integers, the coefficients $A(n;k_1,\ldots,k_s),\ B_{N_1,\ K_1},\ldots,B_{N_r,\ K_r}\ (n_j,\ k_j,\ N_j,\ K_i \geqslant 0,$ $i=1,\ldots,r\ ;\ j=1,\ldots,s)$ are arbitrary constants, real or complex.

$$L^{-1}\left\{(p^2+a^2)^{1/2}\ \{p+(p^2+a^2)^{1/2}\}^{-\lambda}\right.$$

$$,S_n^{m_1,\ldots,m_s}\left[y_1\{p+(p^2+a^2)^{1/2}\}^{-\rho_1},\ldots,y_s\ \{p+(p^2+a^2)^{1/2}\}^{-\rho_s}\right]$$

$$\left.\sum_{i=1}^{r}\left\{\{p+(p^2+b_i^2)^{1/2}\}^{-\nu_i}\ .S_{N_i}^{m_i}\left[z_i\{p+(p^2+b_i^2)^{1/2}\}^{-\sigma_i}\right]\right\}\ ;\ x\right\}$$

$$=\left(\frac{x}{2}\right)^{\lambda+\sum_{i=1}^{r}\nu_i}\ \sum_{\substack{m_1k_1+\ldots+m_sk_s\leqslant n\\ k_1,\ldots,k_s=0}}\ \sum_{K_1=0}^{[N_1/M_1]}\ \ldots\ \sum_{K_r=0}^{[N_r/M_r]}$$

$$.(-n)_{m_1k_1+\ldots+m_sk_s}\ A(n;k_1,\ldots,k_s)\ \frac{y_1^{k_1}}{k_1!}\ldots\frac{y_s^{k_s}}{k_s!}\sum_{i=1}^{r}\left\{(-N_i)_{M_iK_i}B_{N_i,K_i}\frac{z_i^{K_i}}{K_i!}\right\}$$

$$.\frac{\left(\frac{x}{2}\right)^{\sum_{j=1}^{s}\rho_jk_j+\sum_{i=0}^{r}\sigma_iK_i}}{\Gamma(1+\lambda+\sum_{j=1}^{s}\rho_jk_j+\sum_{i=1}^{r}(\nu_i+\sigma_iK_i)}\ F\begin{matrix}0:2;\ldots;2\\ 2:1;\ldots;1\end{matrix}\Bigg[$$

$$---\qquad :$$

$$\tfrac{1}{2}(\lambda+2+\sum_{j=1}^{s}\rho_jk_j+\sum_{i=1}^{r}(\nu_i+\sigma_iK_i)),\ \tfrac{1}{2}(\lambda+1+\sum_{j=1}^{s}\rho_jk_j+\sum_{i=1}^{r}(\nu_i+\sigma_iK_i))\ :$$

$$\tfrac{1}{2}(\lambda+2+\sum_{j=1}^{s}\rho_jk_j),\ \tfrac{1}{2}(\lambda+1+\sum_{j=1}^{s}\rho_jk_j):\tfrac{1}{2}(\nu_1+\sigma_1K_1),\ \tfrac{1}{2}(\nu_1+\sigma_1K_1+1)\ ;\ldots\ ;$$

$$\lambda+1+\sum_{j=1}^{s}\rho_jk_j\qquad :\nu_1+\sigma_1k_1+1\qquad ;\ldots;$$

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$$\frac{1}{2}(\nu_r+\sigma_r K_r),\ \frac{1}{2}(\nu_r+\sigma_r K_r+1)\ ;\quad \frac{-a^2x^2}{4},\ \frac{-b_1^2x^2}{4},\ldots,\ \frac{-b_r^2x^2}{4}\Bigg]. \quad \ldots(3.2)$$
$$\nu_r+\sigma_r K_r+1\ ;$$

provided that

(i) $Re(p) > 0,\ \lambda,\ \rho_i,\ \nu_i, \sigma_i \geqslant 0,\ i=1,\ldots,r;\ j,\ldots,s$

(ii) $m_1,\ldots,m_s,\ M_1,\ldots,M_r$ are arbitrary positive integers, $A(n;\ k_1,\ldots,k_s),\ B_{N_i,\ K_i}\ (n,\ k_1,\ldots,k_s,\ N_iK_i \geqslant 0,\ i,\ldots,r)$ are arbitrary constants, real or complex.

$$L^{-1}\left\{(p)^{\sum_{i=1}^{r}(m_it_i)-\nu}\ (q)^{\sum_{i=1}^{r}(n_is_i)-\mu}\ \sum_{i=1}^{r}\left\{(p^{m_i}\ q^{n_i}+a)^{s_i+t_i}\right\}\right.$$
$$\left. S_N^M\left[z\,p^{-\rho}\,q^{-\sigma}\right];\ x,\ y\right\}$$

$$= (x)^{\sum_{i=1}^{r}(m_it_i)+\nu-1}\ (x)^{\sum_{i=1}^{r}(n_is_i)+\mu-1}\ \sum_{K=0}^{[N/M]}(-N)_{MK}\ B_{N,K}\ \frac{z^K}{K!}\cdot$$

$$F\begin{matrix}0:1;\ldots;1\\2:0;\ldots;0\end{matrix}\Bigg[\begin{matrix}---\\(\sum_{i=1}^{r}(m_is_i)+\nu+\rho K;\ m_1,\ldots,m_r),\ \sum_{i=1}^{r}(n_it_i)+\mu+\sigma K;n_1,\ldots,n_r):\end{matrix}$$

$$\begin{matrix}(s_1+t_1,\ 1);\ldots;(s_r+t_r,\ 1)\ ;\\---\quad;\ldots;\quad---\end{matrix}\quad -ax^{m_1}y^{n_1},\ \ldots,\ ax^{m_r}y^{n_r}\Bigg]. \quad \ldots(3.3)$$

where $S[x_1,\ldots,x_r]$ stands for the generalized Lauricella function.

The conditions of validity of (3.3) are

(i) $\nu,\ \mu,\ \rho,\ \sigma > 0$, $m_i,\ n_i,\ s_i,\ t_i$ are non-negative integers $(i = 1,\ldots,r)$ and $Re(p),\ Re(q),\ Re(a) \geqslant 0$.

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(ii) M is an arbitrary positive integer and the coefficients $B_{N,K}$ $(N, K \geqslant 0)$ are arbitrary constants, real or complex.

Proof of (3.1). On using the convolution theorem for the transform pair (1.1) and (2.1), and evaluating the integral so obtained, we find that

$$L^{-1}\left\{ p^{-2\lambda}(p^2+a^2)^{-\nu}(p^2+b_1^2)^{-(2\mu_1+1)/2}(p^2+b_1^2)^{-(2\mu_2+1)/2} \right.$$

$$\cdot S_{N_1}^{M_1}\left[z_1(p^2+b_1^2)^{-\eta_1}\right] S_{N_2}^{M_2}\left[z_2(p^2+b_2^2)^{-\eta_2}\right]$$

$$\left. \cdot S_{n}^{m_1,\ldots,m_s}\left[y_1 p^{-2\rho_1}(p^2+a^2)^{-\sigma_1},\ldots, y_s p^{-2\rho_s}(p^2+a^2)^{-\sigma_s}\right]; x \right\}$$

$$= x^{2\lambda+2\nu+2\mu_1+2\mu_2+1} \sum_{K_1=0}^{[N_1/M_1]} \sum_{K_2=0}^{[N_2/M_2]} \sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s \leqslant n}$$

$$(-n)_{m_1k_1+\ldots+m_sk_s}$$

$$\cdot (-n_1)_{M_1K_1}(-N_2)_{M_2K_2} A(n;k_1,\ldots,k_s) B_{N_1,K_1}, B_{N_2,K_2}$$

$$\frac{y_1^{k_1}\ldots y_s^{k}}{k!\ldots k_s!} \frac{z_1^{k_1} z_2^{k_2}}{K_1!\ K_2!} \cdot$$

$$\frac{(x)^{\sum_{j=1}^{s}(\rho_j+\sigma_j)k_j+2\eta_1K_1+2\eta_2K_2}}{\Gamma(2\lambda+2\nu+2\mu_1+2\mu_2+2\sum_{j=1}^{s}(\rho_j+\sigma_j)k_j+2\eta_1K_1+2\eta_2K_2+2)}$$

$$\cdot F^{0:1;1;1}_{2:0;0;0}\left[\begin{matrix} ---- \\ \lambda+\nu+\mu_1+\mu_2+\sum_{j=1}^{s}(\rho_j+\sigma_j)k_1+\eta_1K_1+\eta_2K_2+1, \end{matrix}\right.$$

$$\begin{matrix} ---- & : \nu+\sum_{j=1}^{s}\sigma_jk_j \ ; \\ \lambda+\nu+\mu_1+\mu_2+\sum_{j=1}^{s}(\rho_j+\sigma_j)k_j+\eta_1K_1+\eta_2K_2+3/2 : & ---- \ ; \end{matrix}$$

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$$\mu_1+\eta_1 K_1+\tfrac{1}{2}\ ;\ \mu_2+\eta_2 K_2+\tfrac{1}{2}\ ;\ \frac{-a^2x^2}{4}\ ,\ \frac{-b_1^2x^2}{4}\ ,\ \frac{-b_2^2x^2}{4}\ \Bigg] \qquad \ldots(3.4)$$

Now repeating the above process r times, we finally arrive at the required result (3.1)

Proof of (3.2). The proof of (3.2) is similar to that of the result (3.1). Here we use the (1.3) aed (2.2), instead of (1.1) and (2.1).

Proof of (3.3). To prove (3.2), we use the convolution theorem of Laplace transforms for the pair (1.8) and (2.3), and evaluating the u, v-double integral so obtained, we get

$$L^{-1}\left\{(p)^{m_1t_1+m_2t_2-\nu}\ (q)^{m_1s_1+n_2s_2-\mu}\ (p^{m_1}\ q^{n_1}+a)^{-s_1-t_1}\right.$$

$$\left.\cdot\ (p^{m_2}\ q^{n_2}+a)^{-s_2-t_2}\ S_N^M\left[z\,p^{-\rho}\,q^{-\sigma}\right]\ ;\ x,\ y\right\}$$

$$=(x)^{m_1s_1+m_2s_2+\nu-1}\ (y)^{n_1t_1+n_2t_2+\mu-1}\sum_{K=0}^{[N/M]}(-N)_{MK}\ A_{K,N}\ \frac{z^K}{K!}$$

$$\cdot F\begin{matrix}0:1;1\\2:0;0\end{matrix}\left[\begin{matrix}----\ :\\(m_1s_1+m_2s_2+\nu+\rho K;m_1,m_2),\ (n_1y_1+n_2t_2+\mu+\sigma K;n_1,n_2)\ :\end{matrix}\right.$$

$$\left.\begin{matrix}(s_1+t_1,\ 1)\ ;\ (s_2+t_2,1)\ ;\\---\ ;\ ----\ ;\end{matrix}\ -ax^{m_1}y^{n_1}\ ,\ -ax^{m_2}y^{n_2}\right] \qquad \ldots(3.5)$$

Now repeating the above process r times, we immediately get the required result (3.3).

4. SPECIAL CASES

The formulas established in § 2 and § 3 are quite general in character; these involve a product of general classes of polynomials of one and several variables, which can be reduced to a number of known polynomials as mentioned in the papers [4], [5] and [6].

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Acknowledgements

The authors are thankful to Professor H. M. Srivastava (Unixersity of victoria, Canada) for his useful suggestions for the improvement of the paper. The first author (S. P. G.) is also thankful to the University Grants Commission of India for generous financial assisiance.

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Jñānābha. Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

RIGHT ALTERNATIVE RINGS WITH COMMUTATORS IN THE MIDDLE NUCLEUS

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(*Received : June 2, 1992*)

ABSTRACT

Let R be a right alternative ring with characteristic $\neq 2$ and with commutators in the middle nucleus. We show that if R is left primitive, then it is either associative or simple with right identity element. As a by-product we prove that if R is simple, not associative then it has no proper left ideals generated by all alternators

1. INTRODUCTION

Using the standard associator notation $(x,y\,z) = (xy)z - x(yz)$, a right alternative ring is a non-associative ring which satisfies the identity

(i) $(x,y,z) = 0.$

Also, it is known [3] that a right alternative ring with characteristic $\neq 2$ satisfies the following identity :

(1′) $(x,y,z) + (x,z,y) = 0.$

In a right alternative ring R the middle nucleus M is defined as the set

$$M = \{z \in R : (x,y,z) = 0 \text{ for all } x,y \in R\}.$$

Throughout this paper, we assume R to be a right alternative ring with characteristic $\neq 2$ such that

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(2) $(x,y,(z,w)) = 0$

for all x,y,z,w in R where the commutator $(z,w) = zw-wz$.

That is, $(R,R) \subseteq M$.

A straightforward verification shows that any ring satisfies

(T) $(wx,y,z) - (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z$

which is known as the Teichmüller identity.

2. CONSTRUCTION OF IDEALS

Lemma . For any ring R, $(R,R,R) + R(R,R,R)$ is a two-sided ideal of R.

This is proved in [2].

An alternator is an associator of the form (x,x,y), (x,y,x), or (y,x,x).

Lemma 2. Let A be a left ideal of R generated by all alternators. Then

(a) $S = \{s \in A: sR \subseteq A\}$ is a two-sided ideal of R.

(b) $(R,R,A) \subseteq S$.

Proof. (a) This is Theorem 1(a) of [1]

(b) Let $w,x,z \in R$ and $a \in A$. Using (T)

$$(w,x,a)z = (wx,a,z) - (w,xa,z) + (w,x,az) - w(x,a,z)$$

$$= -(wx,z,a) + (w,z,xa) + (w,x,za) - w(x,z,a) \in A$$

(Using (1′) and (2)). Since $(R,R,A) \subseteq A$, we have $(R,R\,A) \subseteq S$.

Theorem 1. If R has a maximal left ideal $A \neq (0)$ generated by all alternators which contains no two sided ideal of R other than (0), then R is associative.

Proof. By Lemma 2, S is a two-sided ideal of R contained in A. Hence $S = (0)$. Since $(R,R,A) \subseteq S$, we have $(R,R,A) = 0$.

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$A+AR$ is clearly a left ideal by *Eq.* (1′). Since A contains all alternators, $(A,R,R) \subset A$ and we see that $A+AR$ is a right ideal as well. Thus $A+AR$ is a two-sided ided of R. Since $A \subset A+AR$, we must have $A+AR = R$. Hence $\neq$

$$(R,R,R) = (R,R,A+AR)$$
$$\subseteq (R,R,A) + (R,R,AR)$$
$$\subseteq (R,R,A) + (R,R,RA) \quad \text{(using (2))}$$
$$\subseteq (R,R,A) = (0).$$

Hence R is associative.

Theorem 2. If R is a simple and not associative, then R has no proper left ideal generated by all alternators.

Proof. Suppose, for example that R has a left ideal $A \neq (0)$ generated by all alternators. We shall show that $A = R$. Now as in the proof of Theorem 1, $A+AR$ is a two-sided ideal of R and $(0) \neq A \subseteq A+AR$ implies that $A+AR \neq (0)$ and since R is a simple ring, we have $A+AR = R$. Then from (2), we have

$$(R,R,R) = (R,R,A=AR) = (R,R,A) + (R,R,AR)$$
$$\subseteq A+(R,R,RA) \subseteq A.$$

Now by Lemma 1, $(R,R,R) + R(R,R,R)$ is a two-sided of R. Also since R is not associative we have that

$(R,R,R) \neq (0)$ and $(0) \neq (R,R,R) \subseteq (R,R,R) + R(R,R,R)$

implies that $(R,R,R) + R(R,R,R)$ is a non-zero ideal of the simple ring R and, therefore, $(R,R,R) + (R,R,R) = R$. Thus

$$R = (R,R,R) + R(R,R,R) \subseteq A+RA \subseteq A.$$

Therefore $A = R$.

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3. LEFT PRIMITIVE RINGS

Definition 1. A left ideal A of R is clled regular if there exists an element $g \in R$ such that $x - xg \in A$ for all $x \in R$.

Definition 2. A ring R is called left primitive if it contains a regular maximal left ideal which contains no two sided of R other then (0).

Theorem 3. If R is a left primitive ring, then either R is associative or it is simple with a right identity element.

Proof. Let A be a regular maximal left ideal generated by all alternators which contains no two sided ideal of R other than (0). If $A \neq (0)$, then by Theorem 1, R is associative Thus assume that (0) is maximal implier, using Zorn's Lemma, that R has no proper ideal. Therefore R is simple. Since (0) is regular, there exists $g \in R$ such that $x - xg \in (0)$ for all $x \in R$. That is, $x = xg$ for all $x \in R$ and g is a right identity element. Hence R is simple with right identity element.

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

SOME VARIANTS OF VIGNOLI'S THEOREM ON α-CONTRACTION

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(Received : June 3, 1992)

1. Introduction. The following theorem of *A.* Vignoli [1] may be regarded as a nice extension of the well-known Darbo's fixed point theorem in the case of a ball. Throughout the paper we assume X to be a real Banach space, $B(0,R)$ a ball with centre at 0 and radius R and $\delta B(0, R)$ the boundary of the ball $B(0,R)$.

Theorem 1. Let $T : B(0, R) \to X$ be α-contractive with the constant $k, (0 \leqslant k < 1)$ and let T satisfies the following condition on the boundary of $B(0, R)$:

(a) if $Tx = \beta x$ for some $x \in \delta B(0,R)$ then $\beta \leqslant \mu$ where μ is any real number satisfying the inequality

(b) $0 \leqslant k < 1 - |1 - \mu|$.

Then there exists $x \in B(0,R)$ such that $T(x) = \mu x$.

Remark 1. When $\mu = 1$, Vignoli's theorem reduces to an extension of Darbo's fixed point theorem in the case of a ball, just like S. Reich's [2] extension of Brouwer's fixod point theorem.

As D. K. Bayen and S. K. Chatterjea [3] considered a nice variant of Reich's theorem and A. K. Sarkar and S. K. Chatterjea [4] have recently considered some other variants of Vignoli's theorem for $\mu = 1$, we intend to consider some new variants of Vignoli's theorem for any real number μ subject to a certain condition.

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2. Some new variants of Vignoli's theorem

Theorem 2. Let $T : B(0,R) \to X$ be α-contractive with constant k, $0 \leqslant k < 1$ and T satisfy the following conditions.

(a) if $Tx = \beta x$ for some $x \in \delta B(0,R)$ then $\beta \geqslant \mu$ where μ is any real number satisfying the inequality

(b) $0 \leqslant k + |1+\mu| < 1$.

Then there exists $z \in B(0,R)$ such that $T(z) = -\mu z$.

Proof. We consider the mapping $F : B(0,R) \to X$ defined by $F(x) = Tx + (1+\mu)x$ for $x \in B(0,R)$ and μ satisfies the condition (b) of the theorem. The mapping F is a α-contractive. Indeed, let A be any subset of $B(0,R)$. Then we have

$$\begin{aligned}\alpha(F(A)) &\leqslant \alpha(T(A)) + |1+\mu|\,\alpha(A)\\ &\leqslant k\,\alpha(A) + |1+\mu|\,\alpha(A)\\ &= (k + |1+\mu|)\,\alpha(A)\end{aligned}$$

condition (b) of the theorem implies $0 \leqslant k + |1+\mu| < 1$, so that F is α-contractive with constant $k + |1+\mu|$.

Let $r : X \to B(0,R)$ be a radial retraction. Since r is non-expansive, the composite mapping $roF : B(0,R) \to B(0,R)$ is α-contractive. Thus by Darbo's fixed point theorem there exists $z \in B(0,R)$ such that $z = roF(z)$.

Thus $z = roF(z) = \begin{cases} F(z), \text{ if } \|F(z)\| \leqslant R \\ R\,\dfrac{F(z)}{\|F(z)\|} \text{ if } \|F(z)\| > R. \end{cases}$

We shall now show that z is the fixed point of F, *i. e.* $F(z) = z$. If $z \in \text{Int } B(0,R)$ then $F(z) = z$, otherwise $z = R\,\dfrac{F(z)}{\|F(z)\|}$, *i. e.* $F(z) = \dfrac{\|F(z)\|}{R} z$, would imply that $\|z\| = R$ which contradicts the assumption $\|z\| < R$. Next suppose that $z \in \delta B(0,R)$ and z is not a fixed point of F. Then $\|F(z)\| \geqslant R$ and

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$$z = R \frac{F(z)}{\| F(z) \|}$$

$$\Rightarrow F(z) = \frac{\| F(z) \|}{R} z$$

$$\Rightarrow T(z) + (1+\mu)z = \frac{\| F(z) \|}{R} z$$

$$\Rightarrow T(z) = \left(\frac{\| F(z) \|}{R} - 1 - \mu \right) z$$

$$= \beta z, \text{ where } \beta = \frac{\| F(z) \|}{R} - 1 - \mu$$

Now $\beta = \frac{\| F(z) \|}{R} - 1 - \mu$

$\Rightarrow -\beta = \mu + (1 - \frac{\| F(z) \|}{R}) < \mu$, which contradicts the condition (a) of the theorem. Thus $F(z) = z$ *i. e.* z is a fixed point of F. Hence $F(z) = z$ implies $T(z) = -\mu z$.

Remark 2. When $\mu = -1$, theorem 2 reduces to another extension (b) of the theorem implies $\mu \in (-2,0)$.

Theorem 3. Let $T \cdot B(0,R) \to X$ be α-contractive with the constant k, $0 \leqslant k < 1$ and let T satisfy the following conditions:

(a) $Tx = \beta x$ for some $x \in \text{Int } B(0,R)$ then $\beta \geqslant -\mu$ where μ is any real number satisfying the inequality

(b) $0 \leqslant k + |1 + \mu| < 1$.

Then there exists $z \in B(0,R)$ such that $T(z) = -\mu z$

Proof. Proceeding exactly in the same way as in the proof of theorem 2, we can prove that

$$z = roF(z) = \begin{cases} F(z), \text{ if } \| F(z) \| \leqslant R \\ R \frac{F(z)}{\| F(z) \|} \text{ if } \| F(z) \| > R. \end{cases}$$

We shall now show that z is a fixed point of F *i.e.* $z = F(z)$.

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If $z \in \delta B(0,R)$ then two cases may arise.

(i) $\| F(z) \| \leqslant R$ and

(ii) $\| F(z) \| > R$.

In the case (1) $z = F(z)$ *i.e.* z is the fixed point of F. But in the case (ii) we have $z = R \dfrac{F(z)}{\| F(z) \|}$ which would imply $F(z) = \dfrac{\| F(z) \| z}{R}$ and no conclusion regarding the fixed point of F can be drawn.

If $z \in Int\ B(0,R)$ then $z = F(z)$, otherwise $z = R \dfrac{F(z)}{\| F(z) \|}$ which would imply $F(z) = \dfrac{\| F(z) \|}{R} z$

$$\Rightarrow (1+\mu+\beta)z = \frac{\| (1+\mu+\beta)z \|}{R}$$

$$= \frac{|(1+\mu+\beta)| \| z \|}{R}$$

$\Rightarrow z \dfrac{\| z \|}{R} z$, which contradicts the assumption $\| z \| < R$.

Therefore $F(z) = z$ and hence $T(z) = -\mu z$.

Remark 3. When $\mu = -1$, theorem 3 reduces to another extension of Darbo's fixed point theorem in the case of a ball like theorem 2. Also condition (b) of theorem 3 implies that $\mu \in (-2,0)$.

Acknowledgements

The author wishes to thank Prof. S. K. Chatterjea, Department of Pure Mathematics, Calcutta University for his kind help during the preparation of this note.

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

SURJECTIVITY OF WEIGHTED Φ -ACCRETIVE OPERATORS

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ABSTRACT

In this paper we consider various tvpes of weighted forms of locally ϕ -accretive operators, and obtain surjectivity results including those of Browder [1, 2], Park and Park [5], Park and Park [6] and Ray [7].

1. INTRODUCTION

Let X be Banach spaces with Y^* the dual of Y, and let $\Phi : X \to Y^+$ be a map satisfying the following:

(1) $\Phi(X)$ is dense in Y^*, and

(2) for such $x \in X$ and each $\alpha \geqslant 0$, $\| \Phi(x) \| \leqslant \| x \|$

and $\Phi(\alpha x) = \alpha \Phi(x)$

A map $P : X \to Y$ is said to be strongly Φ -accretive [1] if there exists a constant $c > 0$ such that, for all $u, v \in X$,

(3) $< Pu - Pv, \Phi(u-v) > \geqslant c \| u-v \|^2$.

The Φ -accretive maps were introduced in an effort to unify the theories for monotone maps (*when* $Y = X^*$) and for accretive maps (when $Y = X$). Surjectivity of those maps was studied by Browder [1, 2], Kirk [4] and Ray [7].

A map $P : X \to Y$ is said to be locally strongly Φ -accretive [4], if, for each $y \in Y$ and $r > 0$, there exists a constant $c > 0$, there exists a constant $c > 0$ such that

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(4) if $\| Px-y \| \leqslant r$ then, for all $u \in X$ sufficiently near to x,

$$< Pu-Px, \Phi(u-x) > \geqslant c \| x-u \|^2.$$

Kirk [4] and Ray [7] extended the surjectivity theorem of Browder [2] to the class of locally strongly Φ -accretive maps by applying the Caristi-Kirk fixed point theorem [3, 4].

For a Banach space Y, we denote by J the duality map from Y into 2^{Y^*} given by

$$J(y) = \{y^* \in Y^* \mid \| y^* \|^2 = \| y \|^2 = < y, y^* >\}$$

It is well known that, by the Hahn-Banach theorem, $J(Y)$ is not empty for each $y \in Y$, J is single valued whenever Y^* is strictly convex, and J is uniformly continuous on bounded subsets of Y whenever Y^* is uniformly convex. Note that for any $y^* \in J(y)$ we have

(5) $\| y \|^2 \leqslant \| z \|^2 = 2 < z-y, y^* >$ for any $z \in Y$

Lemma. [5]. For any $y \in Y$, $y \neq 0$, $y^* \in J(y)$, and $\epsilon > 0$ there exists an $h \in X$ such that $\| h \| \geqslant 1$ and

$$\| \Phi(h) - y^* \| y \|^{-1} \| < \epsilon$$

The duality map J is said to be lower semicontinuous if the following condition holds:

(6) If $\lim_{n \to \infty} y_n = y$ and $y^* \in J(y)$, then there exists a sequence $\{y_n^*\}$ such that

$$y_n^* \in J(y_n) \text{ and } \lim_{n \to \infty} y_n^* = y^*$$

Theorem. Let X and Y be Banach spaces and $P : X \to Y$ a locally Lipschitzian and locally strongly Φ -accretive map. If the duality map J of Y is lower semicontinuous and $P(X)$ is closed, then $P(X) = Y$.

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Motivated by the work [6] of Park and Park, in the present paper the constant C in definitions of (locally) stronglg Φ -accretive maps is replaced by a certain functional value, and we show that some known surjectivity theorems on (locally) strongly Φ -accretive maps can be extended to this general setting.

2. Results. Let $C : [0, \infty) \to (0, \infty)$ be a continuous nonincreasing function.

Let us consider the following types of weighted (locally) Φ -accretive maps.

(i) For anv $u, v \in X$,

$$< Pu-Pv, \Phi(u-v) \geqslant c(\| u-v \| \| h\|^{-1}) \| u-v\|^2$$

(ii) For anv $x \in X$,there exists an $\epsilon > 0$ such that for any $u \in \bar{B}(x;\epsilon)$

$$< Pu-Px, \Phi(u-x) \geqslant c(\| u-x \| \| h\|^{-1}) \| n-x\|^2$$

(iii) For any $x \in X$, there exists an $\epsilon > 0$ such that for any $u, v \in \bar{B}(x; \epsilon)$

$$< Pu-Pv, \Phi(u-v) > \geqslant c(max \| u \|, \| v \| \| h\|^{-1}) \| u-v\|^2$$

(iv) For anv $x \in X$, there exists and $\epsilon > 0$ such that for any $u \in \hat{A}(x; \epsilon)$,

$$< Pu-Px, \Phi(u-x) > \geqslant c(\| x \| \| h\|^{-1}) \| u-x\|^2.$$

Moreover, we state the following type of weighted (locally) Φ -accretive maps.

(v) For each $y \in Y$ and $r > 0$ there exists a non-increasing function $c : [0, \infty) \to (0, \infty)$ such that if $\| Px-y \| \leqslant r$ then for all $u, v \in X$ sufficiently near to x.

$$< Pu-Pv, \Phi(u-v) > \geqslant c(\| u-v \| \| h\|^{-1}) \| u-v\|^2$$

(vi) For each $y \in Y$ and $r > 0$ there exists a non-increasing function

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$c : [0, \infty) \to (0, \infty)$ such that if $\| Px - y \| \leqslant r$ then for all $u \in X$ sufficiently near to x.

$$< Pu - Px, \Phi(u - x) > \geqslant c(\| x \| \| h \|^{-1}) \| u - x \|^2$$

Note that (i) $\Rightarrow$ (ii) $\Rightarrow$ (v) $\Rightarrow$ (vi) and (iii) $\Rightarrow$ (v) $\Rightarrow$ (vii).

Further more, if P is locally strongly Φ -accretive, then P satisfies (vi). Now we have our main result :

Theorem 1. Let X and Y be Banach spaces and $P : X \to Y$ be a locally Lipschitzian map satisfying condition (vi). If the duality map J of X is l. s. c. and $P(x)$ is closed, then $P(x) = Y$

Proof. It suffices to prove that $P(x)$ is open. Since $J(y) \neq \Phi$ for each $y \in Y$, we can choose $y^* \in J(y)$. Since P is locally Lipschitzian for any given $x_0 \in X$, we can choose ϵ_1 such that P is Lipschitzian with a constant M on $\bar{B}(x_0, 2\epsilon_1)$. By (vi), we choose a non-increasing function c and an $\epsilon_2 > 0$ such that, for all

$Px \in \bar{B}(Px_0, 2M\epsilon_1)$ and $u \in \bar{B}(x_0, 2\epsilon_2)$,

$$< Px - Pu, \Phi(x - u) > \geqslant c(\| x \| \| h \|^{-1}) \| x - u \|^2$$

Let $r = \min (c \| x_0 \| \| h \|^{-1} . \epsilon/2, M\epsilon)$, where $\epsilon = \min (\epsilon_1, \epsilon_2)$. Let $a = c(\| x_0 \| \| h \|^{-1}) > 0$. Suppose there exists $y \in \bar{B}(Px_0, r)$ and $y \notin P(x)$. Then we get a contradiction by applying the Caristi-Kirk fixed point theorem [3].

Now we let $d = dist(y, P(x)) > 0$ and $D = \{x \in \bar{B}(x_0, \epsilon) \mid \| y - Px \| \leqslant r\}$. Since $x_0 \in D$, D is a nonempty complete metric space. For any given $x \in D$, by the lemma, we can choose a $h \in X$, $\| h \| \geqslant 1$, such that

$$\| \Phi(h) - (y - Px)^* \| y - Px \|^{-1} \leqslant a/2M.$$

Since $\| Px - Px_0 \| \leqslant \| Px - y \| + \| y - Px_0 \| \leqslant 2r \leqslant 2M\epsilon_1$

We have

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$$< Px_t - Px, \Phi(x_t - x) > \geqslant c(\| x \| \| h \|^{-1}) \| x_t - x \|^2$$

for sufficiently small $t > 0$, where $x_t = x + th$.

Since c is non-increasing,

$$< Px_t - Px, \Phi(x_t - x) > \geqslant c(\| x_0 \| \| h \|^{-1} \| x_t - x \|^2,$$

which implies

$$< Px_t - Px, \Phi(h) > \geqslant at \| h \|^2.$$

Since P is Lipschitzian with constant M on $B(x_0; 2\epsilon_1)$ for small $t > 0$, we have

$$< Px_t - Px, \Phi(h) > \geqslant \frac{a}{M} \| Px_t - Px \|.$$

Now, again applying the lemma,

$$< Px_t - Px, (y - Px)^* >$$

$$= < Px_t - Px, \| y - Px \| \Phi(h) - \| y - Px \| \Phi(h) + (y - Px)^* \geqslant$$

$$\geqslant < Px_t - Px, \Phi(h) > \| y - Px \| - < Px_t - Px, \Phi(h) - (y - Px)^* (y - Px)^{-1} > \| y - Px \|$$

$$\geqslant \frac{a}{m} \| y - Px \| \| Px_t - Px \| - \| y - Px \| \| Px_t - Px \| \| \Phi(h) - (y - Px)^* (y - Px)^{-1} \|$$

$$\geqslant \frac{a}{M} \| y - Px \| Px_t - Px \| - \frac{a}{2M} \| y - Px \| \| Px_t - Px \|.$$

With this inequality and (5), we estimate $\| y - Px_t \|$

$$\| y - Px_t \|^2 \leqslant \| y - Px \|^2 - 2 < Px_t - Px, (y - Px_t)^* >$$

$$= \| y - Px \|^2 - 2 < Px_t - Px, (y - Px)^* - (y - Px)^* + (y - Px_t)^* >$$

$$\leqslant \| y - Px \|^2 - \frac{ad}{M} \| Px_t - Px \| + 2 \| Px_t - Px \| (y - Px)^* - (y - Px_t)^* >$$

Since the duality map J is *l. s. c.* we can select $t > 0$ such that

$$\| (y - Px)^* - (y - Px_t)^* \| \leqslant ad/4M.$$

Therefore,

$$\| y - Px_t \|^2 \leqslant \| y - Px \|^2 - \frac{ad}{2M} \| Px_t - Px \|.$$

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Hence $Px_t \in \bar{B}(y; r)$ and $x_t \in B(x_0; 2\epsilon_1)$. So,

$$\| x_t - x_0 \| \leqslant \frac{1}{c(\| x_0 \| \ \| h \|^{-1})} \| Px_t - Px_0 \|$$

$$\leqslant \frac{1}{c(\| x_0 \|)} (\| Px_t - y \| + \| y - Px_0 \|)$$

$$\leqslant \frac{2r}{c(\| x_0 \|)}$$

i. e. $x_t \in \bar{B}(x_0, \epsilon)$ and $x_t \in D$. Furthermore by (7), $\| Px_t - Px \| \geqslant a \| x_t - x \|$. If we let $g : D \to D$ such that $g(x) = x_t \in D$ and $\Psi(x) = (2m/a^2 d) \| y - Px \|^2$ then we have $\| gx - x \| \leqslant \Psi(x) - \Psi(g(x))$ for $x \in D$. Since Ψ is the continuous map from the complete metric space D into the non-negative reals. Hence, by the Caristi-Kirk fixed point theorem, g has a fixed point in D. Since $\| x_t - x \| = t \| h \| \neq 0$ this is a contradiction. When $\| h \| = 1$, we obtain the following

Corollary: Let X and Y be Banach space $P : X \to Y$ be a locally Lipschitzian map satisfying conditio

(vi)' $< Pu - Px, \Phi(u - x) > \geqslant c(\| x \|) \| u - x \|^2$.

If the duality map J of X is *l. s. c.* and $P(X)$ is closed, then $P(X) = Y$.

Remark. If we replace (vi) with (v) in Theorem 1, then we may obtain the same conclusion. Note also that Theorem 1 extends results of ([1], [2], [5], [6], [7]).

Acknowledgements

The authors are thankful to the U.G.C., New Delhi for providing the financial support to the project.

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

AN EXTENSION OF BROUWER FIXED POINT THEOREM FOR CONTRACTIVE MAPS

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(*Received : October 30, 1992*)

ABSTRACT

This note attempts to extend two recent fixed point theorems for contractive maps of Bridges, Richman, Julian and Mines [*J. Math. Anal, Appl.* 165 (1992), No. 2, 438–456].

1. INTRODUCTION

Let (X,d) be a metric space and $S : X \to X$. Then S is contractive (also called shrinking) if $d(Sx, Sy) < d(x, y)$ for all distinct points x, y of X. It is well known that (see [1], [6]) a contractive map on a compact metric space has a unique fixed point. In view of this classical fixed point theorem of Edelstein [2], Smart [6, *p.* 39] raised the following question: Does every contractive map of the closed unit ball in a Banach space have a fixed point? This question has been negatively answered by *V.* Totik [7]. Indeed, he proved the following

Theorem 1. 1 [7]. There exist a Banach space B and an affine shrinking map S of the closed unit ball U of B into the boundary of U such that S does not have any fixed point.

In view of the above, the following fixed point theorem of D. S. Bridges *et al.* [1] is remarkable.

Theorem 1. 2 [1]. A bounded contractive self-map of R^n has a unique fixed point.

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This leads to the following (see [1], *p.* 441) :

Theorem 1. 3 (Brouwer fixed point theorem for contractive maps). A contractive self-map of a compact convex subset of R^n has a unique fixed point.

The purpose of this paper is to extend Theorems 1.2 and 1. 3 for a pair of maps on R^n.

2. Fixed Point Theorems

Two maps $S, T : X \to X$ are said to have a coincidence if there exists a point z in X such that $Sz = Tz$. Let $C(S, T) = \{z : Sz = Tz\}$. Following Goebel [3] (see also [4], [5]), first we establish an extension of Theorem 1. 2.

Theorem 2.1. Let $S, T : R^n \to R^n$ be such that S is bounded and

$$S(R^n) \subset T(R^n). \tag{1}$$

Suppose that for every $x, y \in R^n$,

$$\left.\begin{array}{l} \| Sx - Sy \| \leqslant \| Tx - Ty \| \text{ if } Tx \neq Ty \\ Sx = Sy \text{ if } Tx = Ty. \end{array}\right\} \tag{2}$$

Then S and T have a conicidence. Further, if S and T commute at $z \in C(S, T)$, than S and T have a uniqne common fixed point.

Proof. Define $F : T(R^n) \to T(R^n)$ by $Fa = S(T^{-1}a)$

for each $a \in T(R^n)$ where T^{-1} denotes the inverse image of a under T. To see that F is a properly defined map, observed that for $x \in T^{-1}a$,

$$Fa = \{x : x \in T^{-1}a\} \subset S(R^n), \tag{3}$$

that is, by (1), $Fa \subset T(R^n)$ for every $a \in T(R^n)$. Further, suppose $b, c \in Fa$. Then there exist $x, y \in T^{-1}a$ such that $b = Sx$ and $c = Sy$. Then (2) implies $b = c$. Thus F is a well defined map on $T(R^n)$. Moreover, by (3), F is bounded, since the image of F is contained in $S(R^n)$ and S is bounded.

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Now, if $a \neq b$, $a, b \in T(R^n)$, then $T^{-1}a \cap T^{-1}b = \phi$.

So $Tx \neq Ty$ for $x \in T^{-1}a$ and $y \in T^{-1}b$.

Thus

$$\| Fa-Fb \| = \| Sx-Sy \| \leq \| Tx-Ty \| = \| a-b \|.$$

Hence, by Theorem 1. 2, F has a unique fixed point, *i. e.*, there exists $u \in T(R^n)$ such that $u=Fu$. If $z \in T^{-1}u$ then

$$Sz=S(T^{-1}u) = Fu=u=Tz.$$

If S and T commute at z, then $SSz=STz=TSz$ If $Tz \neq TSz$, then

$$\| Tz-TSz \| = \| Sz-SSz \| < \| Tz-TSz \|,$$

a contradiction. This proves that $Tz(=Sz)$ is a common fixed point of S and T. The uniqueness of the common fixed point follows from (2).

The following is an extension of Theorem 1. 3.

Theorem 2. 2. Let K be a subset of R^n, and $S, T : K \to K$ such that $T(K)$ is compact and convex. If, for every $x, y \in K$, (2) holds, then S and T have a coincidence. Further, if $STz=TSz$ for every z in $C(S, T)$, then S and T have a unique common fixed point

Proof. An appropriate blend of the proof of Theorem 2. 1 will yield the result

In case $K=R^n$ in Theorem 2 2, we may have the following, which is also a variant of Naimpally *et al* [4, *Cor* 3].

Theorem 2 3. Let (X, d) be a metric space and $S, T : X \to X$ such that $S(X) \subset T(X)$ If $T(X)$ is compact and $d(Sx, Sy) < d(Tx, Ty)$, $Tx \neq Ty$, for $x, y \in X$, then S and T have a coincidence. Further, if $STz=TSz$ for every $z \in C(S, T)$, then S and T have a unique common fixed point.

In [4], the coincidence part of Theorem 2.3 is obtained when S is a multivalued map. One may ask if Theorems 2.1 and 2.2 can be exte-

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nded to the case when S is a multivalued map. In particular, we have the following :

Question. Let $CL(R^n)$ denotes the set of nonempty closed subsets of R^n, H the (generalized) Hausdorff metric on $CL(R^n)$ induced by the norm of R^n, and $S:R^n \to CL(R^n)$, $T:R^n \to R^n$ such that S is bounded and $S(R^n) \subset T(R^n)$. Suppose that for every $x, y \in R^n$,

$$H(Sx, Ty) < \| Tx - Ty \| \text{ if } Tx \neq Ty$$

$$Sx = Sy \quad \text{if} \quad Tx = Ty.$$

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

A GENERAL FIXED POINT THEOREM FOR EXPANSION MAPPINGS

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1. During past fifteen years fixed and common fixed point theorems were established for certain expansion mappings ([1], [2], [3]). In the present paper, we shall prove the following more general theorem which includes the above mentioned results as special cases :

Theorem. Let f, g be surjective selfmaps of a complete metric space (X, d). Suppose there exists a function p, q, r satisfying

(1. 1) $\inf_{x,y X} \{q(x, y) + r(x, y)\} > 1$

(1. 2) $\inf_{x,y X} \{1-p(x, y)\ q(x, y) + r(x, y)\} > 0$

(1. 3) $\sup_{x,y X} \{p(x, y) + q(x, y)\} < 1$ *and*

(1. 4) $$d(fx, gy) \geqslant p(x, y)\ \frac{d(x, fx)\ d(y, fy)}{d(x, y)} + q(x, y)\ \frac{d(y, gy)\ [1 + d(x, f(x))]}{[1 + d(x, y)]} + r(x, y)\ d(x, y)$$

for all x, y in X, $x \neq y$. Then f and g have common fixed point in X.

Proof. Let x_0 be in X. Since f is surjective, there exists a point x_1 in $f^{-1}x_0$, similarly g is surjective, there exists a point x_2 in $g^{-1}x_1$. Continuing we obtain a sequence $\{x_n\}$ with x_{2n+1} in $f^{-1}x_{2n}$, x_{2n+2} in $g^{-1}x_{2n+1}$.

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Suppose $x_{2n+1} = x_{2n}$ for some n. If $x_{2n+1} \neq x_{2n+2}$, then from (1. 4) we write

$$(x_{2n+1}, x_{2n}) = d(gx_{2n+2}, fx_{2n+1}) = d(fx_{2n+1}, gx_{2n+2})$$

$$\geqslant p(x, y) \frac{d(x_{2n+1}, x_{2n})\, d(x_{2n+2}, x_{2n+1})}{d(x_{2n+1}, x_{2n+2})}$$

$$+ q(x, y) \frac{d(x_{2n+2}, x_{2n+1})\,[1 + d(x_{2n+1}, x_{2n})]}{1 + d(x_{2n+1}, x_{2n+2})}$$

$$+ r(x, y)\, d(x_{2n+1}, x_{2n+2}).$$

Thus

$$0 \geqslant (q + r)\, d(x_{2n+1}, x_{2n+2})$$

where q and r are evaluated at $(x, y) = (x_{2n+1}, x_{2n+2})$.

By (1. 2), $d(x_{2n+1}, x_{2n+2}) = 0$.

The condition $x_{2n+1} = x_{2n} = fx_{2n+1}$ implies that x_{2n+1} is a fixed point of f. Also $x_{2n+2} = x_{2n+1} = gx_{2n+2}$ implies x_{2n+2} is fixed point of g. Similarly $x_{2n+2} = x_{2n+1}$ implies $x_{2n+2} = gx_{2n+2}$ and $fx_{2n+1} = x_{2n+1}$ leads to x_{2n+1} as a common fixed point of f and g.

Again from (1. 4), we write

$$d(x_{2n+1}, x_{2n+2}) = d(gx_{2n+2}, fx_{2n+3}) = d(fx_{2n+3}, gx_{2n+2})$$

$$\geqslant p(x, y) \frac{d(x_{2n+3}, x_{2n+2})\, d(x_{2n+2}, x_{2n+1})}{d(x_{2n+3}, x_{2n+2})}$$

$$+ q(x, y) \frac{d(x_{2n+2}, x_{2n+1})\,[1 + d(x_{2n+3}, x_{2n+2})]}{[1 + d(x_{2n+3}, x_{2n+2})]}$$

$$+ r(x, y)\, d(x_{2n+2}, x_{2n+2}).$$

Therefore, $(1-p-q)\, d(x_{2n+1}, x_{2n+2}) \geqslant r d(x_{2n+2}, x_{2n+3})$,

where p, q, r are evaluated at (x_{2n+3}, x_{2n+2}).

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Thus $\{x_n\}$ is a Cauchy sequence, hence is convergent to some x in X. Without loss of generality we may assume that $x_n = x$ for infinitely many n otherewise f and g have a common fixed point. If there exists an infinite number of integers n such that $x_{2n} \neq x$,

then for $y \in g^{-1}x$.

$$d(x_{2n}, x) = d(fx_{2n+1}, gy)$$

$$\geqslant p \frac{d(x_{2n+1}, x_{2n})\, d(y, gy)}{d(x_{2n+1}, y)}$$

$$+ q \frac{d(y, gy)\,[1 + d(x_{2n+1}, x_{2n})]}{[1+d(x_{2n+1}, y)}$$

$$+ rd(x_{2n+1}, y)$$

where p, q, r are evaluated at (x_{2n+1}, y). Thus

$$d(x_{2n}, x) \geqslant (q+r)\ min\ \{dx, y), d(x_{2n+1}, y)\}.$$

Therefore by (1. 1), $x = y$.

Thus if $x_{2n+1} \neq x$ for large n, then $x_{2n} = fx_{2n+1} = x$.

As $n \to \infty$, x is a fixed point of f.

If $x_{2n+1} \neq x$ for imfinitely many n, define $f^{-1}x = z$, and if p, q, r are evaluated at (z, x_{2n+2}) then by (1.4),

$$d(x_{2n+1}, x) = d(gx_{2n+2}\ fz) = d(fz, gx_{2n+2})$$

$$\geqslant p \frac{d(z, fz)\, d(x_{2n+2}, x_{2n+1})}{d(z, x_{2n+2})}$$

$$+ q \frac{d(x_{2n+2}, x_{2n+1})\,[1+d(z, fz)]}{[1+d(z, x_{2n+2})]}$$

$$+ r\, d(z, x_{2n+2})$$

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$$\geqslant \inf_{n,y \; X} (q+r) \min \{d(z, x), d(z, x_{2n+2})\}$$

As $n \to \infty$, therefore an appeal to (1.1) implies $x=z$.

This completes the proof of the theorem.

Remark 1. For $p=q=0$, $r=a$, our theorem reduces to the theorem of Rhoades [3].

Remark 2. For $p=q=0$, $r=a$ and $f=g$, our theorem reduces to the theorem Iseki, Li and Gao [2].

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Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

AN EXPANSION FORMULA FOR MULTIVARIRBLE *H*-FUNCTION INVOLVING GENERALIZED LEGENDRE'S ASSOCIATED FUNCTION

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(*Received* : *October 25, 1992*)

ABSTRACT

The authors establish a new expansion formula for multivariable *H*-function due to Srivastava and Panda [13] in terms of a series of products of the multivariable *H*-function and the generalized Legendre's associated function due to Meulenbeld [9]. A result given earlier by Anandani [2] follows as a special case.

1. INTRODUCTION

The multivariable *H*-function is defined by means of the following *r*-tuple contour integral [14, *p*, 251, *Eq.* (*c.* 1)]:

$$(1.1)\quad H[z_1,\ldots,z_r] = H^{0,\,n:\,m_1,\,n_1;\ldots;m_r,\,n_r}_{p,\,q:\,p_1,\,q_1;\ldots;p_r,\,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix} (a_j;\,\alpha_j',\ldots,\alpha_j^{(r)})_{1,p} \\ (b_j;\,\beta_j',\ldots\beta_j^{(r)})_{1,q} \end{matrix}\right.$$

$$\left.\begin{matrix} (c_j',\,\gamma_j')_{1,q_1};\ldots;(c_j^{(r)},\,\gamma_j^{(r)})_{1,p_r} \\ (d_j',\,\delta_j')_{1,q_1};\ldots;(d_j^{(r)}\,\delta_j^{(r)})_{1,q_r} \end{matrix}\right]$$

$$= [1/(2\pi w)^r]\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)\cdots\phi_r(\xi_r)\,\Psi\,(\xi_1,\ldots,\xi_r)z_1^{\xi_1}\cdots z_r^{\xi_r}$$

$$.\,d\xi_1\ldots d\xi_r,$$

where $w = \sqrt{(-1)}$ and for the definition of the functions $\Psi(\ldots)$ and

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$\phi_i(\xi_i)$ $(i = 1,\ldots,r)$ and as also the condition of existence of the multivariate H-function, we refer to [14, *p.* 251-253, Eqns. $(c.\ 2)$-$(c.8)$].

The various conditions of existence of the multivariate H-function are assumed to be satisfied for multivariate H-function occuring in this paper.

In this paper we evaluate an integral involving generalized associated Legendre function and the multivariate H-function due to Srivastava and Panda and apply it in deriving an expansion for the multivariate H-function in series of products of associated Legendre function and the multivariable H-function.

2. The Integral

The integral to be evaluated is

$$(2.\ 1)\quad \int_{-1}^{1}(1-x)^{\rho-u/2}\,(1+x)^{\sigma+v/2}\,P^{u,v}_{k-(u-v)/2}(x)$$

$$\cdot\, H[(1-x)^{\alpha_1}(1-x)^{\beta_1}\ z_1,\ldots,(1-x)^{\alpha_r}(1+x)^{\beta_r}\ z_r]\ dx$$

$$= 2^{\rho-u+v+\sigma+1}\sum_{t=0}^{\infty}\frac{(-k)_t\,(v-u+k+1)_t}{\Gamma(1-u+t)\ t!}$$

$$\cdot\, H^{0,\,n+2:m_1,\,n_1;\ldots;m_r,\,n_r}_{p+2,\,q+1:p_1,\,q_1;\ldots;p_r,\,q_r}\left[\begin{matrix}2^{\alpha_1}+\beta_1 z_1\\ \vdots\\ 2^{\alpha_r}+\beta_r z_r\end{matrix}\middle|\begin{matrix}(-\sigma-v;\beta_1,\ldots,\beta_r);\\ (b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,q}:\end{matrix}\right.$$

$$\begin{matrix}(u-\rho-t;\alpha_1,\ldots,\alpha_r);\ (a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,p}\ :\ (c_j',\gamma_j')_{1,p_1};\ldots;\\ (u-v-\rho-\sigma-t-1;\ \alpha_1+\beta_1,\ldots,\alpha_r+\beta_r)\ :\ (d_j',\delta_j')_{1,q_1};\ldots;\\ (c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\ (d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\Bigg].$$

The integral (2. 1) is valid under the following set of conditions:

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(i) $(\beta_i, \alpha_i) > 0$; $\forall i \in \{1,\ldots,r\}$; $k - \frac{u-v}{2}$ is a positive integer, k is an integer $\geqslant 0$;

(ii) $Re(\rho - u + \sum_{i=1}^{r} \alpha_i \frac{d_j^{(i)}}{\delta_j^{(i)}} > -1$; $Re(\sigma + v + \sum_{i=1}^{r} \beta_i \frac{d_j^{(i)}}{\delta_j^{(i)}}) > -1$;

$(j = 1,\ldots,m_i;\ i = 1,\ldots,r)$

and the conditions given in [14, *p.* 252-253; Eqns. (*c.* 4), (*c.* 5) and (*c.* 6)] are also satisfied.

Proof. On expressing the multivariable H-function in the integrand as a multiple Mellin-Barnes type integral (1.1) and inverting the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, the value of the integral

$$= (2\pi w)^{-r} \int_{L_1} \cdots \int_{L_r} \Psi(s_1,\ldots,s_r) \sum_{i=1}^{r} \left\{ \phi_i(s_i)\, z_i^{\xi_i} \right\}$$

$$\cdot \left\{ \int_{-1}^{1} (1-x)^{\rho - u/2 + \sum_{i=1}^{r} \alpha_i \xi_i} (1+x)^{\sigma + v/2 + \sum_{i=1}^{r} \beta_i \xi_i} \right.$$

$$\left. \cdot P_{k-(u-v)/2}^{u,v}(x)\, dx \right\} d\xi_1 \ldots d\xi_r.$$

On evaluating the x-integral with the help of the integral [10, *p.* 343, *Eq.* (38)]:

$$(2.\ 2)\quad \int_{-1}^{1} (1-x)^{\rho} (1-x)^{\sigma} P_{k-(m-n)/2}^{m,\,n}(x)\, dx$$

$$= \frac{2^{\rho+\sigma-(m-n)/2}\, \Gamma(\rho - \frac{m}{2} + 1)\, \Gamma(\sigma + \frac{n}{2} + 1)}{\Gamma(1-m)\, \Gamma(\rho + \sigma - \frac{m-n}{2} + 2)}$$

$$\cdot\, {}_3F_2\left(-k,\ n-m+k+1,\ \rho - \frac{m}{2} + 1;\ 1-m,\ \rho - \sigma - \frac{m-n}{2} + 2;\ 1\right),$$

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provided that $Re(\rho - \frac{m}{2}) > -1$, $Re(\sigma + \frac{n}{2}) > -1$; and interpreting the result with the help of (1.1), the integral (2.1) is established.

3. Expansion Theorem

Let the following conditions be satisfied:

(i) $\beta_1,\ldots,\beta_r > 0$; $\alpha_1,\ldots,\alpha_r \geqslant 0$ (or $\beta_1,\ldots,\beta_r \geqslant 0$; $\alpha_1,\ldots,\alpha_r > 0$);

(ii) n, p, q, m_i, n_i, p_i and q_i are integers such that

$0 \leqslant n \leqslant p, q \geqslant 0$, $1 \leqslant m_i \leqslant q_i$ and $0 \leqslant n_i \leqslant p_i$,

$\forall\ i \in \{1,\ldots,r\}$, and the conditions given by [14, *p.* 252–253, Eqns. (*c.* 4), (*c.* 5) and (*c.* 6)] also satisfied.

(iii) $Re(v) > -1$, $Re(u) > 1$, $Re(\rho - u + \sum_{i=1}^{r} \alpha_i \frac{d_j^{(i)}}{\delta_j^{(i)}}) > -1$;

$$Re(\sigma + v + \sum_{i=1}^{r} \beta_i \frac{d_j^{(i)}}{\delta_j^{(i)}}) > -1,\ (j = 1,\ldots,m_i;\ i = 1,\ldots,r).$$

Then the following expansion formula holds:

$$(3.1)\quad (1-x)^{\rho-u/2}(1+x)^{\sigma+v/2}\ H[(1+x)^{\alpha_1}(1+x)^{\beta_1} \cdot z_1,\ldots,(1-x)^{\alpha_r}(1+x)^{\beta_r} z_r]$$

$$= 2^{\rho+\sigma} \sum_{N=0}^{\infty} \sum_{\mu=0}^{N} \frac{(2N-u+v+1)\,\Gamma(N-u+1)\,\Gamma(1+v-u+N+\mu)\,(-N)_\mu}{N!\,\mu!\,\Gamma(1+v+N)\,\Gamma(1-u+\mu)}$$

$$P^{u,\,v}_{N-(u-v)/2}(x) \cdot H^{0,\,n+2:\,m_1,\,n_1;\ldots;m_r,\,n_r}_{p+2,\,q+1:p_1,\,q_1;\ldots;p_r,\,q_r}\left[\begin{matrix} 2^{\alpha_1+\beta_1} z_1 \\ \vdots \\ 2^{\alpha_r+\beta_r} z_n \end{matrix}\right|$$

$$(-v-\sigma;\beta_1,\ldots,\beta_r);\ (u-\rho-\mu;\alpha_1,\ldots,\alpha_r),\ (a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,p};(c_j',\gamma_j')_{1,p_1};$$

$$(b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,q};\ (-1-\rho-\sigma-\mu+u-v;\ \alpha_1+\beta_1,\ldots,\alpha_r+\beta_r):(d_j',\delta_j')_{1,q_1}$$

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$$;\ldots;\ (c_j^{(r)},\ \gamma_j^{(r)})_{1,\,p_r}\ ;\ldots;\ (d_j^{(r)},\ \delta^{(r)})_{1,\,q_r}\ \Big].$$

Proof. Let

$$(3.2)\quad f(x) = (1-x)^{\rho-u/2}\,(1+x)^{\sigma+v/2}\,H[\,(1-x)^{\alpha_1}(1+x)^{\beta_1}z_1;\ldots;\ (1-x)^{\alpha_r}\,(1+x)^{\beta_r}\,z_r\,]$$

$$= \sum_{N=0}^{\infty} C_N\,P^{u,\,v}_{N-(u-v)/2}(x)$$

Equation (3. 2) is valid since $f(x)$ is continuous and of bounded variation in the interval (-1, 1).

Now, multiplying both the sides of (3. 2) by $P^{u,\,v}_{k-(u-v)/2}(x)$ and integrating with respect to x from -1 to 1; evaluating the L. H. S. with the help of (2. 1) and on the R. H. S. interchanging the order of summation, using [4, *p.* 176, Eq. (75)] and then applying orthogonality property of the generalized Legendre's associated functions [10, *p.* 340, Eq. (26) and Eq. (27)]:

$$(3.3)\quad \int_{-1}^{1} P^{u,\,v}_{k-(u-v)/2}(x)\ P^{u,\,v}_{N-(u-v)/2}(x)\,dx$$

$$= \begin{cases} 0, \text{ if } k \neq N \\ \dfrac{2^{u-v+1}\,k!\,\Gamma(k+v+1)}{(2k-u+v+1)\,\Gamma(k-u+1)\,\Gamma(k-u+v+1)}\ ;\ \text{if } k = N \end{cases}$$

provided that $Re(u) < 1$, $Re(v) > -1$; we obtain

$$(3.4)\quad C_k = \frac{2^{\rho+\sigma}\,(2k-u+v+1)\,\Gamma(k-u+1)}{k!\,\Gamma(k+v+1)} \sum_{\mu=0}^{k} \frac{(-k)_\mu\,\Gamma(1+k+v-u+\mu)}{\mu!\,\Gamma(1-u+\mu)}$$

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$$\cdot H \begin{matrix} 0,\ n+2: m_1, n_1; \ldots; m_r, n_r \\ p+2,\ q+1: p_1, q_1; \ldots; p_r, q_r \end{matrix} \left[\begin{matrix} 2^{(\alpha_1+\beta_1)} & z_1 \\ \vdots & \\ 2^{(\alpha_r+\beta_r)} & z_r \end{matrix} \middle| \begin{matrix} (-\nu-\sigma; \beta_1, \ldots, \beta_r); \\ (b_j; \beta_j', \ldots, \beta_j^{(r)})_{1,q}; \end{matrix} \right.$$

$$(u-\rho-\mu; \alpha_1, \ldots, \alpha_r); (a_j, \alpha_j', \ldots, \alpha_j^{(r)})_{1,p}:$$

$$\left. \begin{matrix} (-1-\rho-\sigma-\mu+u\ \nu; \alpha_1+\beta_1, \ldots, \alpha_r+\beta_r): \\ (c_j', \gamma_j')_{1,p_1}; \ldots; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ (d_j', \delta_j')_{1,q_1}; \ldots; d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \right].$$

Now on substituting the value of C_k in (3. 2), the result follows.

4. A Special Case

For $n = p = 0, q = 0$, the multivariate H-function breaks up into a product of r H-functions and consequently, (3. 1) reduces to

$$(4.\ 1)\quad (1-x)^{\rho-m/2}(1+x)^{\sigma+n/2} \prod_{i=1}^{r} \left\{ H_{p_i, q_i}^{m_i, n_i} \left[(1-x)^{\alpha_i} (1+x)^{\beta_i} z_i \middle| \begin{matrix} (c_j^{(i)}, \gamma_j^{(i)})_{1,p_i} \\ (d_j^{(i)}, \delta_j^{(i)})_{1,q_i} \end{matrix} \right] \right.$$

$$= 2^{\rho+\sigma} \sum_{N=0}^{\infty} \sum_{\mu=0}^{N} \frac{(2N-u+v+1)\,\Gamma(N-u+1)\,\Gamma(1+v-u+N+\mu)(-N)_\mu}{n!\,\mu!\,\Gamma(1+v+N)\,\Gamma(1-u+\mu)}$$

$$\cdot P_{N-(u-v)/2}^{u,\ v}(x)\, H \begin{matrix} 0,\ 2: m_1, n; \ldots; m_r, n_r \\ 2,\ 1: p_1, q_1; \ldots; p_r, q_r \end{matrix} \left[\begin{matrix} 2^{(\alpha_1+\beta_1)} & z_1 \\ \vdots & \\ 2^{(\alpha_r+\beta_r)} & z_r \end{matrix} \middle| \begin{matrix} (-\nu-\sigma; \\ (-1-\rho-\sigma-\mu \\ \ldots; \beta_1, \ldots, \beta_r);\ (u-\rho-\mu; \alpha_1, \ldots, \alpha_r) \\ +u-v; \alpha_1+\beta_1, \ldots, \alpha_r+\beta_r) \\ :(c_j', \gamma_j')_{1,p_1}; \ldots; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ :(d_j', \delta_j')_{1,q_1}; \ldots; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \right]$$

For $r=1$, (4. 2) gives rise to the result due to Anandani [2].

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Acknowledgements

The authors thank Professor H. M. Srivastava of the University of Victoria, Victoria, Canada for taking interest in this paper.

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2.	**Periodicity of publication**	Annual
3.	**Printer's name**	*Dr. Anil Kumar Gupta*
	Nationality	Indian
	Address	Navin Printing Press, 98, Gopal Ganj Orai–285001, U. P.
4.	**Publisher's name**	*Dr. R. C. Singh Chandel* For Vijnana Parishad of India
	Nationality	Indian
	Address	D. V. Postgraduate College Orai-285001, U. P., India
5.	**Editor's name**	*Dr. R. C. Singh Chandel*
	Nationality	Indian
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6.	**Name and address of the individuals who own the journal and partners or share hoders holding more than one percent of the total capital**	Vijnana Parishad of India, Address : D. V. Postgraduate College Orai-285001, U. P., India

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